Optimal Auctions with Endogenous Entry

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Abstract

We consider a single object, independent private value auction model with entry. Potential bidders are ex ante symmetric and randomize about entry. After entry, each bidder incurs a cost, then learns her private value and a set of signals that may lead to updated beliefs about other entrants’ valuations. It is shown that the Vickrey auction with free entry maximizes the expected revenue. Furthermore, if the information potentially available to bidders after entry is sufficiently rich, then the Vickrey auction, up to its equivalent class, is also the only optimal sealed-bid auction.

Keywords and Phrases: Auctions, Vickrey auction, efficient entry, complete classes.
JEL Classification Numbers: D44, D82.

1 Introduction

The traditional literature on optimal auctions and revenue comparisons generally assumes that there is a fixed set of bidders, and that the bidders are endowed with information about their valuations.¹ In this paper, we relax this assumption by taking entry into account, so that the set of bidders is endogenously determined. Specifically, we consider an auction model in which the object is a single indivisible asset with independent private value (IPV). There is an entry cost for each potential bidder, which can be

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interpreted as the cost of preparing a bid or estimating the valuations of the asset, etc. There are a large number of potential bidders, who randomize about entry so that the ex ante expected profit from attending the auction is zero. Before entry, bidders know nothing except a very rough estimate about the valuation of the asset, which is represented by a common prior (a distribution). After entry, each bidder incurs an entry cost and then learns her private value \( (t_i) \) and in addition, a set of signals \( (s) \) that may lead to updated beliefs (distributions) about entrant bidders' valuations. One typical scenario may proceed as follows: After entry, the entrant bidders have access to extensive information about the asset for sale. By exerting effort and studying the asset closely, each bidder learns her own private value. In addition, simply by knowing information such as other entrant bidders' identities after entry, each bidder may talk to her consulting team or search from the Internet to learn more about her rivals. This may result in more precise estimates about her rivals' valuations. For simplicity we model this situation as a belief updating process. In such a setup, we show that the Vickrey auction (the second-price sealed bid auction) with free entry maximizes the expected revenue. Furthermore, the Vickrey auction, up to its equivalent class, is also the only optimal sealed-bid auction if the information potentially available to bidders after entry is sufficiently rich, in the sense that the signal system can generate a "complete class" of posterior distributions.

Our model extends the existing literature on auctions with costly entry. In earlier work (e.g., Johnson (1979), French and McCormick (1984), McAfee and McMillan (1987), Engelbrecht-Wiggans (1993), Levin and Smith (1994)) entry is usually formulated such that potential bidders do not possess private information, and after incurring an entry cost, each entrant learns a private signal about the value of the object. Our model differs from these formulations in one important respect: while beliefs (distributions about other bidders' valuations) are usually assumed to be fixed both before and after entry occurs, in our model bidders can update their beliefs after entry. We believe that this setup is not only theoretically more general, but also practically more realistic. For example, in some procurement auctions in the construction industry, contestants are usually not certain about their competitors' identities until they get into the final stage, which is a costly process to prepare blueprints or detailed proposals. During this "due diligence" stage, they can in general learn more about their rivals' positions such as their capacities, overhead costs, and even reputations regarding whether they are "tough" or "soft" in the final bid tender, etc. This sort of "belief" updating can greatly enhance the contestants' information before the final auction is conducted.²

²Bid tabs provide company records of who bid what on construction contracts. The bid tabs underlying the data analysis in Dyer and Kagel (1996) demonstrate the difficulty of competing against a particular low overhead rival. Firms
In our analysis, we focus on symmetric entry equilibria, in which each potential bidder randomizes about entry with the same probability. The symmetric entry equilibrium was first introduced by Milgrom (1981) and received thorough analysis in Levin and Smith (1994). In any symmetric entry equilibrium, the expected rent to the bidders is driven down to zero by endogenous entry; hence the expected revenue to the seller is the same as the expected total surplus generated from the sale. This implies that any optimal auction must be efficient, which tremendously simplifies our job in searching for optimal auctions.

The main result of this paper is that we identify an environment under which the optimal auction is unique. Optimal auctions are traditionally characterized by an optimal reserve price and are typically not unique in terms of auction rules, which is also the case when costly entry is taken into account. For example, in the symmetric IPV setting, Levin and Smith (1994) show that the first-price auction and second-price auction can both be optimal if the entry fee and reserve price are set optimally. However, when beliefs can be updated after entry, and the post-entry environment is sufficiently rich, we show in this research that the optimal auction can only be Vickrey auction, up to its equivalent class. The general intuition goes as follows: under the complete class assumption, the information potentially available to bidders after entry is so rich that the final auction is typically an asymmetric one, with a sufficiently large magnitude of asymmetry. To ensure the allocation efficiency, the only mechanism that survives the “environment test” is the Vickrey auction, up to its equivalent class.

Our result also provides a new explanation for the prevalence of simple auctions (English auctions and Vickrey auctions). Neeman (2003) defines a concept of effectiveness as a measure of the proximity to optimality, and shows that the effectiveness of the simple auctions is quite high: while an optimal auction may outperform a simple auction in terms of the expected revenues generated, it may not outperform it by much. In our model, by taking endogenous entry into account, we show that the simple auctions themselves are optimal. This provides another explanation for the prevalence of simple auctions.

In terms of the methodology, the notion of “complete classes” plays a central role in deriving the uniqueness result. We first show that any optimal auction must be (ex post) efficient. We then show that any efficient, incentive compatible (IC) direct revelation mechanism (DRM) is equivalent to a

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Note that in the IPV setting, an English auction is outcome-equivalent, though not strategically equivalent, to a Vickrey auction.
Vickrey-Clarke-Groves (VCG) mechanism from the interim perspective (i.e., at the point in time when each bidder has just learned her private information). Finally, we show that if the (belief) signal system generates a "complete class" for the payment rule space, then the above equivalence also holds in an "almost everywhere" sense. In other words, any optimal mechanism in this environment must be essentially a VCG mechanism.

The paper is organized as follows. Section 2 presents the model. Section 3 shows the optimality of the Vickrey auction with free entry. Section 4 identifies conditions under which the Vickrey auction, up to its equivalent class, is also the unique optimal sealed-bid auction. Section 5 is a discussion and Section 6 concludes.

2 The Model

We consider a mechanism through which a single indivisible asset is offered for sale to $N$ potentially interested buyers. Ex ante, the potential buyers are symmetric and hold a common prior that their private values will be independent draws from a distribution $G(\cdot)$. In contrast to the traditional auction model, we augment the selling mechanism with an entry game in which information acquisition is costly. Suppose that $n$ potential buyers decide to attend the auction. Then after incurring an entry cost $c$, each entrant bidder learns her private value $t_i$, and a signal vector $s = (s_1, \ldots, s_n)$, which, combined with the initial belief $G(\cdot)$, will lead to updated beliefs about entrant bidders' valuations. For example, suppose ex ante that each potential buyer holds a belief that the valuation will be drawn from Uniform$[0, 1]$; after entry, bidder $i$'s belief is updated such that entrant $j$'s value is drawn from Uniform$[0, s_j]$ for all $j \in E \setminus \{i\}$, where $E = \{1, \ldots, n\}$ is the index set of entrant bidders.\footnote{In a recent paper, Fang and Morris (2004) analyze two-bidder auction games with similar information structure, in which each bidder knows her own private value and a noisy signal about the other bidder's value. They characterize equilibria under both first-price and second-price auctions.}

$s$ is a set of public signals known to the entrant bidders. Since each bidder knows her own valuation ($t_i$), it is not essential to assume that she also learns $s_i$. However, for ease of equilibrium analysis, we assume that each bidder ($i$) knows her own signal $s_i$ as well. In this way each bidder also knows what other bidders think about her (in terms of beliefs about her valuation). We assume that the conditional distribution of $S_i$ given the list of other variables $(t_1, \ldots, t_n, s_{-i})$ depends only on $t_i$. (A special case is the standard model in which $S_i = t_i$; that is, the signals precisely identify the underlying types.) Formally, we assume that conditional on $t_i$, $S_i$ is drawn from distribution $F(\cdot|t_i)$. With abuse
of notation, we denote the support of \( S_i \) given \( t_i \) as \( \text{Supp}(F(\cdot|t_i)) \), and the unconditional support of \( S_i \) as \( \mathcal{S}_i \).

For our main results to hold, we do not need to specify the exact structure of belief updating, though Bayesian updating would be a natural way for the formulation. We require only that the signal generating system (denoted as \( \{S\} \)) be consistent, in the sense that it gives rise to the ex ante common prior \( G(\cdot) \).

The seller’s own valuation is normalized to be zero. Bidder \( i \)'s valuation \( t_i \in [0, \bar{t}_i] (=: T_i), \ i \in E \). For notational simplicity, let \( T = T_1 \times \cdots \times T_n \), \( T_{-i} = T_1 \times \cdots \times T_{i-1} \times T_{i+1} \cdots \times T_n \), \( \mathcal{S} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_n \), and \( \mathcal{S}_{-i} = \mathcal{S}_1 \times \cdots \times \mathcal{S}_{i-1} \times \mathcal{S}_{i+1} \times \cdots \times \mathcal{S}_n \).

Both the seller and the buyers are assumed to be risk-neutral. In this paper, we also assume that \( N \) or \( c \) is large enough so that if all potential buyers enter the auction with probability one, their expected profits will be strictly negative. This assumption distinguishes our model from traditional auction models that do not consider entry cost.

At the outset of the game the seller moves first by announcing the selling mechanism (the allocation rule, the payment rule, the reserve price, the entry fee, and so on). Taking the selling mechanism as given, the potential buyers make entry decisions. Finally the entrant bidders submit sealed bids.

Since potential buyers are ex ante symmetric, in the following analysis we will focus exclusively on symmetric entry equilibrium, in which each potential buyer enters the auction with the same probability \( q^* \), where \( q^* \) is determined by \( N, c, G(\cdot) \) and the selling mechanism.\(^5\)

### 3 The Optimality of the Vickrey Auction

Since the selling mechanism must be announced before \( s \) is realized, an optimal auction in this context must maximize the expected revenue regardless of the post-entry environment (or the realization of \( S \)). In this section we establish that the Vickrey auction is one such optimal auction.

**Theorem 1** The Vickrey auction with free entry maximizes the expected revenue.

\(^5\)See Levin and Smith (1994) for the way equilibrium \( q^* \) is determined endogenously by the potential bidders' randomization condition. Note that despite the ex ante symmetry, the potential buyers' entry decision could be asymmetric. For example, there are equilibria where exactly \( n^* \) buyers enter the auction and \( N - n^* \) buyers stay out (see, e.g., Johnson (1979), McAfee and McMillan (1987), and Engelbrecht-Wiggans (1993)). In a recent paper, Smith and Levin (2000) use an experimental approach to test how people would coordinate with entry. Their results strongly reject the hypothesis of asymmetric entry (deterministic entry) and tend to favor the alternative hypothesis that entry is symmetric (stochastic entry).
**Proof:** In any symmetric entry equilibrium, the randomization condition implies that the expected profit to the bidders is zero. Consequently, in equilibrium the seller’s expected revenue is the same as the expected surplus generated from the sale. Hence it suffices to show that the Vickrey auction with free entry generates the maximal possible expected surplus.

Let $S_v$ be the surplus generated by a Vickrey auction, and $S_a$ the surplus generated by an alternative auction. The Vickrey auction is ex post efficient.\(^6\) Therefore, conditional on \(n\), the number of actual bidders, we have

\[
S_v \geq S_a
\]

(1)

Hence,

\[
E(S_v|n) \geq E(S_a|n)
\]

(2)

Note that when each potential buyer randomizes about entry with probability \(q\), the actual number of entrants \(n\) is distributed as Binomial \((N, q)\). Hence inequality (2) implies

\[
E(S_v|q) \geq E(S_a|q) \ \forall q \in (0,1)
\]

(3)

Let \(q^* \in \arg\max_q E(S_v|q)\), then

\[
E(S_v|q^*) \geq E(S_a|q) \ \forall q \in (0,1)
\]

(4)

Following arguments paralleling those in Levin and Smith, we can show that under a Vickrey auction with free entry (zero entry fee), the endogenously determined entry probability (denoted as \(q_v\)) is exactly the same as the unconstrained optimizer \(q^*\). Therefore \(E(S_v|q_v) \geq E(S_a|q) \ \forall q \in (0,1)\). In particular, \(E(S_v|q_v) \geq E(S_a|q_a)\), where \(q_a\) is the equilibrium probability of entry induced by the alternative auction mechanism.

Hence the Vickrey auction with free entry indeed maximizes the expected surplus. *Q.E.D.*

The Vickrey auction does not only achieve ex post efficient allocation, but also induces “first best” entry with zero entry fee (in the sense of achieving unconstrained optimal entry probability \(q^*\)). Therefore it maximizes the expected surplus (and hence the expected revenue) in this costly entry environment.

The optimality of free entry under the Vickrey auction can be understood as follows:

\(^6\)We assume that in equilibrium, each bidder employs the dominant strategy of bidding her true valuation.
Let \( t_{j,n} \) denote the \( j \)th highest order statistic among \( n \) iid valuation samples. In a symmetric IPV setting one can show the following identity:

\[
E(t_{1:n} - t_{1:n-1}) = \frac{1}{n}E(t_{1:n} - t_{2:n})
\]

(5)

(The intuition is that drawing one more value from a distribution and adding it to the previously drawn \( n - 1 \) values will enable this newly drawn value to be the highest one with probability \( 1/n \).)

Now consider the case of deterministic entry and ignore the integer constraint of \( n \) for the moment. Given an entry fee \( e \), bidders will enter the auction until the expected rent is driven down to zero. That is

\[
\frac{1}{n}E(t_{1:n} - t_{2:n}) = c + e
\]

(6)

Substituting Eq. (5) into (6), we have\(^7\)

\[
E(t_{1:n} - t_{1:n-1}) = c + e
\]

(7)

Therefore, when the entry fee \( e = 0 \), the marginal social gain of entry \( E(t_{1:n} - t_{1:n-1}) \) is exactly equalized by the marginal social cost of entry \( (c) \) in equilibrium. This implies that zero entry fee induces efficient entry in the deterministic entry case.

Now when symmetric entry is assumed, the randomization of each bidder’s entry decision serves to “smooth” the integer problem and the above intuition still carries over. So the Vickrey auction with free entry is optimal.

The proof of Theorem 1 also suggests the following Corollary, which will be most helpful for the next section analysis.

**Corollary 1** Any optimal auction must be (ex post) efficient.

4 **The Uniqueness of the Optimal Auction**

By Corollary 1, the (ex post) efficiency is necessary for a mechanism to be optimal. Since beliefs can be updated after entry, the final auction is in general an asymmetric one. This would impose a strong restriction on the class of mechanisms that can induce efficient allocation. For example, under a first-price auction the equilibrium would be very sensitive to the asymmetry among bidders, and the

\(^7\)Note that this substitution is not “exact,” since identity (5) only makes sense when \( n \) is an integer. We are being vague here in order to get a rough intuition.
outcome would often be inefficient. This may narrow down the set of optimal auctions. On the other hand, as is well-known, a Vickrey auction always induces ex post efficient allocation in private value settings. This is true regardless of the asymmetry among bidders. (This is true even when bidders possess inconsistent beliefs about each other’s valuations.) We thus ask the following question: Is the Vickrey auction also the only sealed-bid auction that achieves optimality in this environment? It turns out that we can indeed identify exact conditions under which the Vickrey auction, up to its equivalent class, is the unique optimal sealed-bid auction.

To that end first we need to define what we mean by auctions. In what follows, we shall restrict our search for optimal auctions within the class of standard auctions, which is defined below:

**Definition 1 (Standard auctions:)** Standard auctions are sealed-bid auctions in which: (1) the rules are not contingent on the belief signal \( s \); (2) the bidder with the highest bid wins the object; (3) the bidder with the lowest possible type makes zero expected payment.

The definition of standard auctions here is consistent with the spirit in Wilson (1987), who suggests that the trading rules not depend on particular “environments” (e.g., common prior, bidders’ probability assessments about each other’s private information). Indeed, real world auction rules typically process bids only and rarely involve distributions. In our setting, since \( s \) is not realized when the selling mechanism is announced, it would be even harder for a mechanism to be made contingent on \( s \).

The Revelation Principle implies that to identify the set of social choice functions that are implementable (in dominant strategies or Bayesian Nash equilibrium strategies), we need only identify those

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\(^8\) Usually “weak” bidders would bid more aggressively than “strong” bidders, as shown in Griesmer et al. (1967), Plum (1992), Lebrun (1999), and Maskin and Riley (2000a,b).

\(^9\) Beyond the class of sealed-bid auctions, we know that the Vickrey auction cannot be the unique optimal auction, since the English ascending bid auction is outcome-equivalent to a Vickrey auction.

\(^10\) We would argue that restricting the search for optimal auctions to the standard auctions defined here is without loss of generality; since a mechanism being contingent on \( s \) may not be desirable – it would simply be costly for the auctioneer to figure out \( s \). First, it would be costly for the seller to learn \( s \) directly; Second, though one may argue that the seller can make the bidders to reveal \( s \) and hence learn \( s \) indirectly, in practice there are some relevant costs that favor mechanisms that reveal as little information as possible. For example, there is some small probability of information leakage that would disadvantage a bidder in another setting, say a bargaining environment, in which the other bargainer is uninformed about the value distribution. Then, the seller would bear a cost of forcing information revelation by discouraging entry. In either case above, the auctioneer may find it sub-optimal to make a selling mechanism contingent on \( s \) given the optimality of the Vickrey auction that we have established.
that are truthfully implementable. Without loss of generality, to identify optimal auctions we henceforth focus on the analysis of the incentive compatible (IC) direct revelation mechanism (DRM). By Corollary 1 and Definition 1, we can further restrict our attention to the IC efficient DRM that depends only on the report about $t$. Note that even though the seller chooses the auction format (a standard auction defined in Definition 1), he may not know the induced DRM since he may choose not to learn $s$ in any optimal auction. However, the bidders observe $s$ and hence can infer the induced DRM. For this reason we can still carry out DRM analysis.

Given a report profile on types $t = (t_i, t_{-i})$, a DRM is described by an allocation rule $y(t) = \{y_i(t)\}_{i=1}^n$ and a payment rule $x(t) = \{x_i(t)\}_{i=1}^n$, where $y_i(t) \in [0, 1]$ is bidder $i$'s probability of winning the object, and $x_i(t)$ is the monetary payment that bidder $i$ makes to the seller when $t$ is reported. Write the social choice function $f(t) = (y(t), x(t))$.

In any DRM characterized by allocation rule $y(\cdot)$ and payment rule $x(\cdot)$, if bidder $i$ reports $t'_i$ while all the other bidders report truthfully, then bidder $i$'s payoff (or utility) is

$$U_i(t'_i, t_{-i}; t_i) = u_i(f(t'_i, t_{-i}; t_i) = t_i \cdot y_i(t'_i, t_{-i}) - x_i(t'_i, t_{-i})$$

Define

$$q_i(t'_i|s_{-i}) = E_{t_{-i}}(y_i(t'_i, t_{-i})|S_{-i} = s_{-i}) \quad \text{and} \quad m_i(t'_i|s_{-i}) = E_{t_{-i}}(x_i(t'_i, t_{-i})|S_{-i} = s_{-i})$$

(9)

to be, respectively, bidder $i$'s interim expected probability of winning and interim expected payment by reporting $t'_i$ conditional on $(t_i, s_{-i})$ and everyone else reporting $t_{-i}$ truthfully. Her interim expected payoff is then

$$V_i(t'_i|t_i, s_{-i}) = E_{t_{-i}}(U_i(t'_i, t_{-i}; t_i)|S_{-i} = s_{-i}) = t_i \cdot q_i(t'_i|s_{-i}) - m_i(t'_i|s_{-i})$$

(10)

Let $V_i(t_i|s_{-i})$ be bidder $i$'s interim expected payoff from reporting truthfully when all the other bidders are also reporting truthfully. Incentive compatibility implies

$$V_i(t_i|s_{-i}) = V_i(t_i|t_i, s_{-i}) = \max_{t'_i \in T_i} V_i(t'_i|t_i, s_{-i}) = t_i \cdot q_i(t_i|s_{-i}) - m_i(t_i|s_{-i})$$

(11)
It is easily verified that $V_i(t_i|s_{-i})$ is a convex function which is differentiable almost everywhere. By the Envelope Theorem, we have

$$\frac{dV_i(t_i|s_{-i})}{d t_i} = \frac{\partial V_i(t_i',t_i, s_{-i})}{\partial t_i'}|_{t_i'=t_i} = q_i(t_i|s_{-i}) \text{ a.e. } t_i$$

(12)

The convexity also implies that $V_i(\cdot|s_{-i})$ is absolutely continuous (with respect to the Lebesgue measure). Therefore,

$$V_i(t_i|s_{-i}) = V_i(0|s_{-i}) + \int_0^{t_i} q_i(z|s_{-i}) \, dz$$

(13)

$V_i(0|s_{-i}) = 0$ by the definition of standard auctions. Thus by equations (11) and (13), we have

$$m_i(t_i|s_{-i}) = t_i \cdot q_i(t_i|s_{-i}) - \int_0^{t_i} q_i(z|s_{-i}) \, dz$$

(14)

As can be seen from (14), the interim expected payment rule in any incentive compatible DRM is completely determined by the interim expected allocation rule $q(\cdot)$, which is in turn determined by the allocation rule $x(\cdot)$.

Since any two efficient allocation rules coincide almost everywhere, Eq. (14) implies that any IC efficient DRM must be equivalent to a Vickrey-Clarke-Groves (VCG) mechanism at the point in time when each entrant has just learned $(t_i, s_{-i})$. We thus have the following lemma:

**Lemma 1** Any optimal DRM is payoff-equivalent to a VCG mechanism at the point in time when each entrant has just learned $(t_i, s_{-i})$.

Without the restriction that the lowest possible type must make zero expected profit, the above result holds more generally: from the interim perspective any IC efficient DRM is payoff-equivalent to a Groves’ mechanism (the expected payoff to the lowest type may vary arbitrarily). This equivalence result is shown in Williams (1999) as allowing for more general utility functions.\(^{11}\) In our case, the linear form of the utility function renders us a simple proof.

Lemma 1 implies that under interim expectation, the payment rule of any optimal DRM must be equivalent to the payment rule in a VCG mechanism. We next show that if the signal system is so “rich” that it generates a “complete class,” then the above equivalence result also holds (almost everywhere) after removing the expectations. We formally introduce the definition of complete classes for a functional space $\mathcal{F}$ (defined on $T$) below:

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\(^{11}\)This result is also implied, but not explicitly stated in Mookherjee and Reichelstein (1992).
**Definition 2** (F-Complete classes:) A functional family $\mathcal{H}$ (defined on $T$) is a complete class for a functional space $\mathcal{F}$ if for $f, g \in \mathcal{F}$, that $\int f h = \int g h$ for all $h \in \mathcal{H}$ implies $f = g$ a.e.

As an example, by standard dual space arguments, $L^q$ is a complete class for $L^p$ (or $L^q$ is a $L^p$-complete class), where $1 \leq p < \infty$, and $1/p + 1/q = 1$. In our problem, $\mathcal{F}$ is the payment rule functional space, and $\mathcal{H}$ is a conditional distributional family that is generated from the post-entry signal system (denoted as $\{S\}$). More specifically, let $h(t|s)$ be $t$’s density function conditional on the observed $s$, and let $\{H(t|S)|$ be the family containing all the conditional densities arising from any possible signal $s \in S$. Then a (conditional) distribution family $\{H(t|S)|$ forms a complete class for the payment rule space $\mathcal{F}$ (defined on $T$) if for $f, g \in \mathcal{F}$, that $E(f(t)|s) = E(g(t)|s)$ for all $s \in S$ implies $f(t) = g(t)$ a.e.

Since $t$ is bounded, the feasible payment rules are also bounded. More generally, in this paper we will consider the payment rules that are essentially bounded. Formally, let $x(t)$ and $x'(t)$ be two payment rules under two given mechanisms, then $x, x' \in L^\infty(T)$, where $L^\infty(T)$ is the space of all measurable functions defined on $T$ that are bounded except possibly on a subset of measure zero.

Let $\{S_j\}$ denote the signal system that can generate any possible signal about $t_j \in T_j$, $j \in E$. The following assumption about the post-entry signal system is central for the proof of the uniqueness theorem:

**Assumption (CC):** For each entrant $i \in E$, the signal system $\{S_{-i}\}$ generates a complete class for $L^\infty(T_{-i})$.

Two complete classes in this context are stated as follows:

**Lemma 2** Assumption (CC) is satisfied if a dense collection of uniform distributions on $T_j$ can be induced by $\{S_j\}$, or a dense collection of (truncated) normal distributions on $T_j$ can be induced by $\{S_j\}$, $j \in E \setminus \{i\}$.

**Proof:** See Appendix.

Note that the examples provided in Lemma 2 are sufficient but may not be necessary for generating $L^\infty$-complete classes. More complete classes can be found along the same line as in the proof of Lemma 2. For example, one can show that as long as a signal system can generate distributions that approximate
uniform distributions in $L^1$ on $T$, then such signal system also induces a complete class for $L^\infty$.\footnote{12}

By Assumption (CC) and the interim equivalence result stated in Lemma 1, it is immediate that the payment rule in any optimal DRM must be essentially a VCG payment rule. Before we give a formal proof for the uniqueness result, we need one more definition, that is, the class of Vickrey-equivalent auctions.

**Definition 3 (Vickrey-equivalent auctions:)** An auction is a Vickrey-equivalent auction if it is a dominant strategy mechanism in which the bidder with the highest valuation wins and pays the amount of the second-highest valuation.\footnote{13}

For example, let $\varphi(\cdot)$ be any nonnegative and strictly increasing function, and $b_{2,n}$ be the second highest bid; then any auction in which the bidder with the highest bid wins and pays $\varphi(b_{2,n})$ is a Vickrey-equivalent auction.

The following lemma shows that a Vickrey-equivalent auction is equivalent to a Vickrey auction in the sense of strategic equivalence.

**Lemma 3** Any Vickrey-equivalent auction is strategically equivalent to a Vickrey auction.

**Proof:** See Appendix.

We are finally ready to state the uniqueness result:

**Theorem 2** Under Assumption (CC), any optimal (sealed-bid) auction must be a Vickrey auction, up to its equivalent class.

**Proof:** By Lemma 1, any optimal DRM must be interim equivalent to a VCG mechanism. By Assumption (CC), any optimal DRM must be essentially a VCG mechanism. Therefore any optimal auction must be a dominant strategy mechanism in which the bidder with the highest valuation wins and pays the amount equal to the second highest valuation. By Definition 3 and Lemma 3, any optimal auction must be strategically equivalent to a Vickrey auction. *Q.E.D.*

\footnote{12}{Theoretically speaking we can always identify the dual space as one of the complete classes. However, the dual of $L^\infty$ is (strictly) larger than $L^1$, which does not contain a countable base. Given the difficulty in characterizing such a dual space, we turn to a different approach to look for alternative complete classes, which results in Lemma 2.}

\footnote{13}{By definition, a Vickrey auction is trivially a Vickrey-equivalent auction.}
In view of Theorem 1 and Theorem 2, the Vickrey auction (up to its equivalent class) with free entry, is the unique optimal sealed-bid auction in this costly entry environment consistent with Definitions 1-3 and Assumption (CC).

As is clear by now, the driving force for this uniqueness result is the allocation efficiency. Due to endogenous entry, any optimal auction must induce (ex post) efficient allocation. Since the bidders can update their beliefs after entry so that the final auction is typically an asymmetric one, the only mechanism that guarantees the efficient allocation is the Vickrey auction, up to its equivalent class.

5 Discussion

An alternative proof of the uniqueness result can be constructed by combining the complete class assumption and a result proven in Holmström (1979). First, using a stronger version of the complete class assumption,\(^{14}\) any IC DRM must use dominant strategies. Second, we apply Holmström's result to conclude that any dominant strategy DRM implementing an efficient outcome must be a Groves mechanism. Specifically, Holmström shows that when the agent’s valuation domain is smoothly connected (in particular, convex), then any transfer scheme implementing an efficient outcome in dominant strategies is a Groves scheme. In our case, each bidder’s valuation function does have a convex domain, thus Holmström’s result applies and the optimal DRM must be a Groves mechanism. Note that this alternative proof and the proof given in the previous section are essentially the same in spirit.

Similar to Holmström’s approach, earlier papers by Green and Laffont (1977,1978), and Walker (1978) also identify conditions on the utility domain under which the transfer scheme implementing an efficient outcome in dominant strategies must be a Groves scheme. In this paper, we introduce an alternative approach leading to a Groves mechanism: instead of identifying conditions on the utility domain, we focus on conditions concerning the information system.

Note that although in the preceding analysis we restrict our attention to symmetric entry equilibrium, in the asymmetric entry equilibrium our main results are still valid with a minor modification regarding the entry fee payments. When a deterministic number of bidders enter the auction (in an asymmetric entry equilibrium), entrant bidders would usually end up with positive rent due to the integer problem. However from our previous analysis we can see that as long as an appropriate entry fee is imposed to extract the remaining rent, then the Vickrey auction remains to be optimal, and the uniqueness of

\(^{14}\)That is, \(H\) forms a complete class for \(F\) if \(\forall f, g \in F, \int fh \geq \int gh\) for every \(h \in H, h \geq 0\) implies that \(f \geq g\) a.e. In the appendix we actually show that Lemma 2 holds for this stronger version of complete classes.
optimal auction still holds.

Finally it is worth noting that the bidders’ ex ante symmetry is also crucial for the uniqueness result to hold. A counter-example can be adapted from Landsberger et al. (2001). In their paper, a first-price sealed bid auction with two bidders is analyzed when the ranking of valuations is common knowledge among bidders. They show that there exists an equilibrium in which each bidder bids according to a strictly increasing function, with the lower bidder bidding more aggressively than the higher bidder. As a result, the lower bidder wins the object with some positive probability. Now introduce entry cost into the model so that bidders have to incur a cost $c$ in order to learn their exact valuations, but suppose the ranking of the valuations is still common knowledge before entry. Given the result in Landsberger et al. (2001), under a first-price sealed-bid auction, it is obvious that if the entry cost is not too large, both bidders will still participate in the auction and the expected revenue will be positive. However, if a Vickrey auction is conducted, the lower bidder will never enter as long as the entry cost is positive (no matter how small it is), since she would never win the object and the entry cost could never be compensated in equilibrium.\footnote{Again, assume that both bidders use their dominant strategies and bid their values truthfully.} As a result, the auction will be conducted in the presence of the higher bidder only. This example can be easily extended to the $n$-bidder case. That is, only the highest bidder will enter a Vickrey auction, leading to zero revenue for the seller. In this case, the Vickrey auction is not revenue maximizing, but rather revenue minimizing! This example shows how sensitive our result is to the environment.

6 Conclusion

Optimality and efficiency are central to both auction design and practice. While closely related, optimality and efficiency do not always coincide. Revenue maximization typically involves a trade-off between efficiency and rent extraction, and hence is technically complicated and could even be informationally infeasible. In the symmetric entry equilibrium, these two goals coincide. This greatly simplifies the job faced by a mechanism designer. Since any mechanism achieving optimality must award the good to the bidder with the highest valuation, the mechanism designer can restrict the search for optimal auctions to the class of efficient mechanisms. Taking entry into account appears to complicate the setup at the outset, but the optimal auction design problem turns out to be simpler in the sense that a simple auction itself is optimal.
There are several reasons why the Vickrey auction is appealing. First, a Vickrey auction induces truthful reporting as a dominant strategy, and this is true even when bidders' private valuations are correlated. Second, we can feel fairly confident that a rational bidder under a Vickrey auction will indeed play the dominant strategy. Third, the outcome is ex post efficient in dominant strategy equilibrium under a Vickrey auction. This research presents one more reason to appreciate the Vickrey auction, that is, in the whole class of mechanisms we have discussed, the Vickrey auction is optimal, and is also the only optimal auction that survives the environment test.

Finally, the tool developed in this paper to show the uniqueness of an optimal mechanism is noteworthy. First, we have the interim equivalence result between any efficient incentive compatible DRM and a VCG mechanism. Second, the complete class assumption enables us to conclude that not only should they be equivalent from the interim perspective, they should also be equivalent in an "almost everywhere" sense. Since the interim equivalence result holds in a fairly general environment, and complete classes can be characterized in different contexts, this approach suggests some potential for showing the uniqueness of mechanisms that are both efficient and incentive compatible in other environments.

\[16\text{Due to the (outcome) equivalence between a Vickrey auction and an English auction in private value environment, the following statement also applies to English auctions.}\]
Appendix

Proof of Lemma 2: Let \( \mathcal{H} \) be a family of nonnegative functions on \( T \). That \( \mathcal{H} \) forms an \( L^\infty \)-complete class is equivalent to the following statement: \( \forall f, g \in L^\infty, \int f h = \int g h \) for any \( h \in \mathcal{H} \Rightarrow f = g \ a.e. \) By the linearity of integration, \( \mathcal{H} \) forms an \( L^\infty \)-complete class if \( \int f h = 0 \) for any \( h \in \mathcal{H} \Rightarrow f = 0 \ a.e. \) Since \( h \geq 0 \), to show that \( \mathcal{H} \) is an \( L^\infty \)-complete class, it suffices to show that \( \int f h \geq 0 \) for all \( h \in \mathcal{H} \) implies that \( f \geq 0 \ a.e. \) Let \( d = |E| - 1 \), where \( |E| \) denotes the number of entrant bidders. The proof will be completed by establishing the following five steps.

Step 1: The class (denoted as \( \mathcal{H}1 \)) of all indicator (or characteristic) functions \( 1_A \), where \( A \) is some measurable subset of \( T_{-i} \) and \( 0 < m(A) < \infty \), forms a complete class for \( L^\infty(T_{-i}) \).

Suppose not, then \( \int f h \geq 0 \) for all \( h \in \mathcal{H}1 \), but \( m(\{f < 0\}) > 0 \). Since \( \{f < 0\} = \bigcup_{n=1}^\infty \{f < -\frac{1}{n}\} \), this implies that \( m(\{f < -\frac{1}{n}\}) > 0 \) for some \( n^* \). Now choose a subset \( A \subseteq \{f < -\frac{1}{n}\} \subseteq T_{-i} \), such that \( 0 < m(A) < \infty \). Let \( h = 1_A \). Obviously \( h \in \mathcal{H}1 \), but \( \int f h = \int_A f < -\frac{1}{n} m(A) < 0 \), a contradiction.

Step 2: The class (denoted as \( \mathcal{H}2 \)) of all indicator functions \( 1_B \), where \( B \) is some \( d \)-dimensional “rectangle” in \( T_{-i} \), forms a complete class for \( L^\infty(T_{-i}) \).

Given any indicator function \( 1_A \), where \( A \) is a subset of \( T_{-i} \) defined in Step 1, it can be approximated by step functions on \( T_{-i} \) in \( L^1 \). Hence,

\[
\sum_{i=1}^n c_i 1_{B_i} \rightarrow 1_A \text{ in } L^1
\]  

(15)

for some \( c_i \) and \( B_i \), where \( c_i > 0 \) and \( B_i \)'s are disjoint \( d \)-dimensional “rectangles.”

Therefore,

\[
|\int f \sum c_i 1_{B_i} - \int f 1_A| \leq \int |f(\sum c_i 1_{B_i} - 1_A)|
\]
\[
\leq \|f\|_\infty \cdot \|\sum c_i 1_{B_i} - 1_A\|_1 \text{ (by Hölder's Inequality)}
\]
\[
\rightarrow 0 \text{ (by (15) and that } f \in L^\infty\)
\]

On the other hand, we have

\[
\int f \sum c_i 1_{B_i} = \sum c_i \int f 1_{B_i} \geq 0 \text{ (by hypothesis)}
\]

Therefore, \( \int f 1_A \geq 0 \). Since \( 1_A \) is an arbitrary indicator function in \( \mathcal{H}1 \), we must have \( f \geq 0 \ a.e. \) by the conclusion in Step 1. This implies that \( \mathcal{H}2 \) is a complete class for \( L^\infty \).

Step 3: The class (denoted as \( \mathcal{H}3 \)) of all functions \( \prod_{j \in E \setminus \{i\}} u_j \) on \( T_{-i} \), where \( u_j \) is some uniform density function defined on \( T_j \), is a complete class for \( L^\infty(T_{-i}) \).
This step follows from Step 2 trivially since any function $1_B$ defined in Step 2 can be normalized to be a product of $d$ uniform density functions.

Note that $\mathcal{H}_3$ can do the job as long as it contains a dense collection of the uniform density functions described above. Moreover, in our model, the joint density function on $\mathcal{L}_i$ induced by $\{S_i\}$ is the product of $|E| - 1$ one-dimensional density functions (by conditional independence). This completes the proof for the first part of the claim in Lemma 2.

**Step 4:** Any one-dimensional uniform density function on $T_j$ can be approximated in $L^1$ by a finite mixture of (truncated) normal density functions on $T_j$.

We will first show that the approximation can be achieved by a finite mixture of normal density functions on the real line. The proof of this step is facilitated by the following Lemma:

**Lemma A1** Let $\{h_n\}$ be a sequence of functions in $L^1$ such that $h_n \to h$ a.e. with $h \in L^1$. Then $\int |h_n| \to \int |h|$ implies $\int |h_n - h| \to 0$.

**Proof:** Let $f_n = |h_n - h|$, $f = 0$, $g_n = |h_n| + |h|$, and $g = 2|h|$.

$g, g_n \in L^1$ since $h, h_n \in L^1$. $f_n \leq |h_n| + |h| = g_n$.

$h_n \to h$ a.e. $\Rightarrow g_n \to 2|h| = g$ a.e.

Since $g_n - f_n \geq 0$, $g_n - f_n \to g - f$ a.e., by Fatou’s Lemma,

$$\lim \inf \int g_n - f_n \geq \int g - f$$

$\int g_n \to \int g$ since $\int |h_n| \to \int |h|$, thus we have

$$\int g - \lim \sup \int f_n \geq \int g - \int f$$

Therefore,

$$\lim \sup \int f_n \leq \int f = 0. \quad (16)$$

Similarly, considering $g_n + f_n$, we get

$$\lim \inf \int f_n \geq \int f = 0 \quad (17)$$

The lemma follows by combining equations (16) and (17). Q.E.D.

Based on this lemma, given any uniform density function $h(x) = \frac{1}{|a - b|} 1_{(a,b)}(x)$, where $(a,b) \subset T_j$, we will show that it can be approximated in $L^1$ by the equi-weighted mixture of $n$ normal densities over $R$, with means $\mu_{n,k} = (b - a) \frac{k - 0.5}{n} + a$ for $k = 1, \ldots, n$, and common standard deviation $\sigma_n = \frac{1}{n}$. 

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In view of Lemma A1, it suffices to show that

\[
    h_n(x) = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{2\pi}\sigma_n} \cdot e^{-\frac{(x-\mu_{n,k})^2}{2\sigma_n^2}}
\]

\[
    = \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{n} e^{-\frac{n}{2}[n(x-a)-(b-a)(k-0.5)]^2}
\]

\[
    \rightarrow \frac{1}{b-a} 1_{(a,b)}(x) \text{ a.e. as } n \rightarrow \infty.
\]

This can be shown as follows:

1) When \( x < a \), \([n(x-a)-(b-a)(k-0.5)]^2 > n^2(x-a)^2 \),

\[
    h_n(x) \leq \frac{n}{\sqrt{2\pi}} e^{-\frac{n}{2}|n(x-a)|^2} \rightarrow 0
\]

Therefore, \( h_n(x) \rightarrow 0 \) when \( x < a \);

2) When \( x > b \), let \( x = b + \delta \), where \( \delta > 0 \), then \([n(x-a)-(b-a)(k-0.5)]^2 > n^2\delta^2 \). Analogously to 1) above, \( h_n(x) \rightarrow 0 \) when \( x > b \);

3) When \( x \in (a,b) \),

\[
    h_n(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(b-a)^2} \sum_{k=1}^{n} e^{-\frac{n^2}{2}(x-a)^2} \frac{1}{n} g\left(\frac{k}{n}\right)
\]

\[
    = \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}(b-a)^2} \sum_{k=1}^{n} \frac{1}{n} g\left(\frac{k}{n}\right)
\]

\[
    = \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}(b-a)^2} \left( \int_0^1 g(y) \, dy + O\left(\frac{1}{n^2}\right) \right)
\]

\[
    = \frac{n}{\sqrt{2\pi}} e^{-\frac{1}{2}(b-a)^2} \int_{n(x-a)}^{n(x-b)} e^{-\frac{1}{2}(u^2+(b-a)u)} \left( -\frac{1}{n(b-a)} \right) \, du + O\left(\frac{1}{n}\right)
\]

\[
    = \frac{1}{\sqrt{2\pi}} \frac{1}{b-a} \int_{n(x-a)}^{n(x-b)} e^{-\frac{1}{2}(u^2+(b-a)u)^2} \, du + O\left(\frac{1}{n}\right)
\]

\[
    \rightarrow \frac{1}{b-a} \int_{n(x-a)}^{n(x-b)} e^{-\frac{1}{2}(u^2+(b-a)u)^2} \, du
\]

The fourth equality is obtained by change of variable \( u = n[(x-a)-(b-a)y] \).

Therefore, \( h_n(x) \rightarrow h(x) = \frac{1}{b-a} 1_{(a,b)}(x) \) a.e. By Lemma A1, any uniform density function on \( T_j \) can be approximated in \( L^1 \) with a finite mixture of normal distributions on the real line.

We next show that the approximation can be achieved by a finite mixture of truncated normal density functions on \( T_j \).

Define

\[
    C_{n,k} = \int_{T_j} \frac{1}{\sqrt{2\pi}\sigma_n} \cdot e^{-\frac{(x-\mu_{n,k})^2}{2\sigma_n^2}} \, dx
\]

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Then the truncated normal density function of $N(\mu, \sigma^2)$ on $T_j$ is given by

$$\varphi_{n,k}(x) = \frac{1}{C_{n,k}} \frac{1}{\sqrt{2\pi\sigma_n}} e^{-\frac{(x-\mu)}{2\sigma_n^2}}$$  \hspace{1cm} (18)

Let $C_n = \int_{T_j} h_n(x) \, dx$, then

$$C_n = \frac{1}{n} \sum_{k=1}^{n} C_{n,k}$$

Since $T_j$ is a subset of $R$, the result established above, $\int_R |h_n(x) - h(x)| \, dx \to 0$, implies that $\int_{T_j} |h_n(x) - h(x)| \, dx \to 0$. But this implies that $\int_{T_j} h_n(x) \, dx \to \int_{T_j} h(x) \, dx = 1$, i.e., $C_n \to 1$.

Define new weights $\omega_{n,k} = C_{n,k}/(nC_n)$, and form a mixture of truncated normal densities:

$$g_n(x) = \sum_{k=1}^{n} \omega_{n,k} \varphi_{n,k}(x)$$

$$= \sum_{k=1}^{n} \frac{C_{n,k}}{nC_n} \frac{1}{\sqrt{2\pi\sigma_n}} e^{-\frac{(x-\mu)}{2\sigma_n^2}}$$

$$= \frac{1}{C_n} \cdot h_n(x)$$

Then

$$\int_{T_j} |g_n(x) - h(x)| \leq \int_{T_j} |g_n(x) - h_n(x)| + \int_{T_j} |h_n(x) - h(x)|$$

$$= \frac{1}{C_n} - 1 \int_{T_j} h_n(x) + \int_{T_j} |h_n(x) - h(x)|$$

$$\to 0.$$

This completes the proof for Step 4.

**Step 5:** The class (denoted as $\mathcal{H}_4$) consisting of all functions $\{\prod_{j \in E \setminus \{i\}} \varphi_j(x_j)\}$, where $\varphi_j(\cdot)$ is some truncated normal density function on $T_j$ defined in (18) is a complete class for $L^\infty(T_\setminus i)$.

We show a lemma first.

**Lemma A2** Let $u_j$ be a uniform density function on $T_j$, then given any function $\prod_{j \in E \setminus \{i\}} u_j$, there exists $\{\Psi_{n_j}(x_j)\}_{j \in E \setminus \{i\}}$, where each $\Psi_{n_j}(\cdot)$ is a mixture of $n_j$ truncated normal densities on $T_j$, such that $\prod_{j \in E \setminus \{i\}} \Psi_{n_j}(x_j) \to \prod_{j \in E \setminus \{i\}} u_j(x_j)$ in $L^1$ as $n_j \to \infty$ for all $j \in E \setminus \{i\}$.

**Proof:** Let’s consider the case $d = 2$. As shown in Step 4, given any two one-dimensional uniform density functions $u_1(x_1)$ and $u_2(x_2)$, there exist two finite mixtures of (truncated) normal density functions $\Psi_{n_1}(x_1)$ on $T_1$ and $\Psi_{n_2}(x_2)$ on $T_2$ such that $\|\Psi_{n_j} - u_j\|_1 \to 0$ for $j = 1, 2$.

$$\int (\Psi_{n_1}(x_1)\Psi_{n_2}(x_2) - u_1(x_1)u_2(x_2)) \, dx_1 \, dx_2$$
\[
\begin{align*}
\leq & \int [\Psi_{n_1}(x_1) - u_1(x_1) | \Psi_{n_2}(x_2) + u_1(x_1) | \Psi_{n_2}(x_2) - u_2(x_2) ] \, dx_1 \, dx_2 \\
\leq & \int [\Psi_{n_1}(x_1) - u_1(x_1) | (| \Psi_{n_2}(x_2) - u_2(x_2) | + u_2(x_2)) + u_1(x_1) | \Psi_{n_2}(x_2) - u_2(x_2) ] \, dx_1 \, dx_2 \\
= & \int [\Psi_{n_1}(x_1) - u_1(x_1) ] \, dx_1 \cdot \int [(| \Psi_{n_2}(x_2) - u_2(x_2) | + u_2(x_2)) ] \, dx_2 \\
& + \int u_1(x_1) \, dx_1 \cdot \int [ | \Psi_{n_2}(x_2) - u_2(x_2) ] \, dx_2 \\
\to & 0
\end{align*}
\]

Using the method of mathematical induction, the generalization to an arbitrary \( d \)-dimensional case is straightforward. \( Q.E.D. \)

Now given any \( h \in \mathcal{H} \), by Lemma A2, we can find mixtures of (truncated) normal densities \( \{ \Psi_{n_j}^{(j)} \}_{j \in E \setminus \{i\}} \) such that \( \prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)} \rightarrow h \) in \( L^1 \). By Hölder’s Inequality again, we have

\[
\int f \cdot \prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)} \rightarrow \int f \, h \tag{19}
\]

On the other hand,

\[
\begin{align*}
\int f \cdot \prod_{j \in E \setminus \{i\}} \Psi_{n_j}^{(j)}(x_j) & = \int f \cdot \prod_{j \in E \setminus \{i\}} \left( \sum_{k=1}^{n_j} \omega_{n_j, k} \varphi_{n_j, k}^{(j)} (x_j) \right) \\
& = \int f \cdot \sum_{k=1}^{\prod_{j \in E \setminus \{i\}} n_j} \omega_k \Omega_k \quad \text{(where } \Omega_k \text{ is a product of } d \text{ (truncated) normal densities and } \omega_k \text{ is a product of } d \text{ non-negative weights)} \\
& = \sum_{k=1}^{\prod_{j \in E \setminus \{i\}} n_j} \omega_k \int f \Omega_k \\
& \geq 0 \quad \text{(since } \int f \Omega_k \geq 0 \text{ by hypothesis)} 
\end{align*}
\tag{20}
\]

By (19) and (20),

\[
\int f \, h \geq 0
\]

This holds for arbitrary \( h \in \mathcal{H} \), by the conclusion in Step 3, \( f \geq 0 \) \( a.e. \), which implies that \( \mathcal{H} \) is a complete class for \( L^\infty \).

This completes the proof for the second part of the claim in Lemma 2. \( Q.E.D. \)

**Proof of Lemma 3:** Two auctions are strategically equivalent if there exists an isomorphism between their strategy spaces which preserves payoffs. Our job is to find a one-to-one mapping which is payoff-preserving.
Let \( \hat{\Gamma}(\hat{W}, g(\cdot)) \) be a dominant strategy mechanism (auction) in which the bidder with the highest valuation wins and pays the amount of second highest valuation. \( \hat{W} \) is the strategy set, and \( g(\cdot) \) is the outcome function. Define the social choice function \( f(t) = (y(t), x(t)) \) where \( y(t) \) is an efficient assignment rule and \( x(t) \) is the Vickrey payment scheme. By assumption, there is a dominant strategy \( B^*(\cdot) \) which implements \( f(t) \). We therefore have

\[
g(B^*(t)) = f(t)
\]  

(21)

where \( B^*(t) = (b^*_1(t_1), b^*_2(t_2), \ldots, b^*_n(t_n)) \in \hat{W} \).

Now define strategy set \( W = B^*(T) = (b^*_1(T_1), b^*_2(T_2), \ldots, b^*_n(T_n)) \), we need to show that the mechanism \( \Gamma(W, g(\cdot)) \) is strategically equivalent to a Vickrey auction.

First, Definitions 1 and 3 imply that \( b^*_i(\cdot) \) has to be symmetric and strictly increasing; otherwise the outcome may not be efficient for some realized \( t \). Therefore \( b^*_i(\cdot) = b^*(\cdot) \forall i \in E \) and \( b^*(\cdot) \) is strictly increasing. Hence \( B^*: T \to W \) is a one-to-one mapping.

It remains to show that the mapping \( B^* \) is payoff-preserving. Given a report profile \( \hat{t} = (\hat{t}_i, \hat{t}_{-i}) \in T \), \( i \)’s utility derived from a Vickrey auction can be written as \( V_i(\hat{t}; t_i) = u_i(f(\hat{t}); t_i) \). Given a report profile \( B^*(\hat{t}) \in W \), bidder \( i \)’s utility derived from auction \( \Gamma \) can be written as \( \hat{V}_i(B^*(\hat{t}); t_i) = u_i(g(B^*(\hat{t}); t_i). \) Since

\[
\hat{V}_i(B^*(\hat{t}); t_i) = u_i(g(B^*(\hat{t}); t_i)
\]

\[
= u_i(f(\hat{t}); t_i) \quad \text{(by (21))}
\]

\[
= V_i(\hat{t}; t_i),
\]

the one-to-one mapping \( B^* \) is indeed payoff-preserving. \( Q.E.D. \)
References


