1. (30 points)

A decision maker (DM) faces a situation in which there are four possible consequences (or outcomes), \( C = \{1, 2, 3, 4\} \). The DM has preferences over lotteries over consequences, and these preferences satisfy consequentialism, the continuity axiom, and the independence axiom.

We will denote a simple lottery over consequences by \((\pi_1, \pi_2, \pi_3, \pi_4)\), where \(\pi_k\) denotes the probability that the lottery assigns to consequence \(k\). Suppose that the DM’s preferences are represented by the utility function \(U(p_1, p_2, p_3, p_4)\), where we have

\[
egin{align*}
U(1, 0, 0, 0) &= 1 \\
U(0, 1, 0, 0) &= 2 \\
U(0, 0, 1, 0) &= 3 \\
U(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) &= 4.
\end{align*}
\]

Based on the above information, compute \(U(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10})\). Show and explain your work.

Answer:

Because preferences satisfy consequentialism, continuity, and independence, we know that they must be represented by a von Neumann-Morgenstern utility function of the form

\[U(p_1, p_2, p_3, p_4) = p_1 u_1 + p_2 u_2 + p_3 u_3 + p_4 u_4.\]

From \(U(1, 0, 0, 0) = 1\), we conclude that \(u_1 = 1\), from \(U(0, 1, 0, 0) = 2\), we conclude that \(u_2 = 2\), and from \(U(0, 0, 1, 0) = 3\), we conclude that \(u_3 = 3\). Therefore, we have

\[4 = U(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = \frac{1}{4} \cdot 1 + \frac{1}{4} \cdot 2 + \frac{1}{4} \cdot 3 + \frac{1}{4} \cdot u_4.\]

Solving for \(u_4\), we have

\[
\begin{align*}
4 - \frac{1}{4} - \frac{2}{4} - \frac{3}{4} &= \frac{1}{4} \cdot u_4 \\
u_4 &= 10.
\end{align*}
\]
Therefore,

\[ U(\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}) = \frac{1}{10} \cdot 1 + \frac{2}{10} \cdot 2 + \frac{3}{10} \cdot 3 + \frac{4}{10} \cdot 10 = \frac{54}{10} \]

2. (30 points)

The following problem concerns a pure exchange economy with \( k \) goods and \( n \) consumers. You can assume that each consumer’s utility function is strictly quasi-concave, strictly monotonic, and continuous. Consider an allocation \( x^* = (x^*_1, ..., x^*_n) >> 0 \) that is weakly Pareto optimal.

**(Prove that there is at least one consumer who weakly prefers her bundle to the bundle of any other consumer. That is, prove that there exists a consumer \( i \) such that**

\[ u_i(x^*_i) \geq u_i(x^*_h) \]

**for all \( h = 1, ..., n \).**

**(Answer:**

I will offer two proofs, but you should pick one proof and make that your answer.

**Direct proof:** Since the weakly PO allocation is strictly interior and utility functions satisfy continuity, strict quasi-concavity, and strict monotonicity, we can apply the SFTWE to conclude that there exists \( p^* \) such that \( (p^*, x^*) \) is a CE. In the CE, the budget constraint of consumer \( h \) is \( p^* \cdot x_h \leq p^* \cdot x^*_h \). There must be at least one consumer \( i \) whose income (the right side of her budget constraint) is at least as high as any other consumer’s income. Thus, consumer \( i \) could afford to purchase the bundle of any consumer. Since \( x^*_i \) solves her utility maximization problem, the conclusion follows.

**Proof by contradiction:** Suppose that the conclusion is false. Then for all \( i \) there exists \( h \) such that \( u_i(x^*_h) > u_i(x^*_i) \). It will be convenient to let the function \( g \) select another consumer whose bundle is preferred to a given consumer’s bundle. Thus, for consumer \( g(i) \) we have \( u_i(x^*_g(i)) > u_i(x^*_i) \). Start with consumer 1 and consider consumer \( g(1) \). If \( g(g(1)) = 1 \), then consumers 1 and \( g(1) \) form a cycle that starts and ends with consumer 1 (more on that later). If \( g(1) \) does not form a cycle then consider consumer \( g(g(1)) \). If \( g(g(g(1))) = 1 \) then consumers 1, \( g(1) \), and \( g(g(1)) \) form a cycle that starts and ends with consumer 1. If \( g(g(g(1))) = g(1) \) then consumers \( g(1) \) and \( g(g(1)) \) form a cycle that starts and ends with consumer \( g(1) \). If \( g(g(1)) \) does not form a cycle, then we bring in a new consumer, \( g(g(g(1))) \). Since there are only \( n \) consumers, eventually we must bring in a new consumer that prefers a bundle of an existing consumer, thereby completing a cycle.

Once we have a cycle, consider the new allocation in which each consumer \( i \) in the cycle receives \( x^*_{g(i)} \), and each consumer \( i \) not in the cycle receives \( x^*_i \). Clearly this allocation is feasible. Each consumer in the cycle strictly prefers her
new bundle, so $x^*$ is not strongly PO. Because utility functions satisfy continuity and strict monotonicity, the set of SPO and WPO allocations coincide, so $x^*$ is not weakly PO, a contradiction.

3. (40 points)
Consider the following pure-exchange economy with two types of consumers and two goods. There are $n_1$ type-1 consumers and $n_2$ type-2 consumers. If consumer $i$ is of type 1, she has the utility function

$$u_i(x^1_i, x^2_i) = \log(x^1_i) + \log(x^2_i)$$

and the initial endowment vector, $(1, 0)$. If consumer $i$ is of type 2, she has the utility function

$$u_i(x^1_i, x^2_i) = \log(x^1_i) + \log(x^2_i)$$

and the initial endowment vector, $(0, 1)$.

(a) (10 points) Define a competitive equilibrium for this economy.
(b) (20 points) Compute the competitive equilibrium price and allocation.
(c) (10 points) If the ratio $n_1/n_2$ increases, does the utility received by type-1 consumers at the CE go up, go down, or remain the same? Explain your reasoning.

Answer:
(a) A CE is a price, $(p^{1*}, p^{2*})$ and an allocation, $(x^{1*}_i, x^{2*}_i)_{i=1}^{n_1+n_2}$, such that

(i) for type-1 consumers, $(x^{1*}_i, x^{2*}_i)$ solves

$$\max \log(x^1_i) + \log(x^2_i)$$
subject to

$$p^{1*} x^1_i + p^{2*} x^2_i \leq p^{1*}$$
$$x_i \geq 0,$$

(ii) for type-2 consumers, $(x^{1*}_i, x^{2*}_i)$ solves

$$\max \log(x^1_i) + \log(x^2_i)$$
subject to

$$p^{1*} x^1_i + p^{2*} x^2_i \leq p^{2*}$$
$$x_i \geq 0,$$

(iii) markets clear:

$$\sum_{i=1}^{n_1+n_2} x^1_i \leq n_1$$
$$\sum_{i=1}^{n_1+n_2} x^2_i \leq n_2.$$
(b) Because the utility functions are strictly monotonic, budget inequalities and resource inequalities will hold as equalities. By strict quasi-concavity, utility maximization problems have unique solutions, so I will denote the consumption of type-1 consumers as $x_1$ and the consumption of type-2 consumers as $x_2$. Also, I will normalize the price of good 2 to be 1 and denote the price of good 1 as $p$.

For type-1 consumers, the demand function is found by solving the budget equation and the MRS condition, $x_2^2/x_1^1 = p$, yielding

$$x_1^1 = \frac{1}{2}$$
$$x_1^2 = \frac{p}{2}$$

For type-2 consumers, the demand function is found by solving the budget equation and the MRS condition, $x_2^2/x_1^2 = p$, yielding

$$x_2^1 = \frac{1}{2p}$$
$$x_2^2 = \frac{1}{2}$$

The market clearing condition for good 1 is then given by

$$n_1(\frac{1}{2}) + n_2(\frac{1}{2p}) = n_1.$$ 

Solving for $p$, we have

$$p = \frac{n_2}{n_1}.$$ 

Substituting the equilibrium price into the demand functions, we have the equilibrium allocation

$$x_1^1 = \frac{1}{2}$$
$$x_1^2 = \frac{n_2}{2n_1}$$
$$x_2^1 = \frac{n_1}{2n_2}$$
$$x_2^2 = \frac{1}{2}.$$ 

(c) At the CE, the utility of type-1 consumers is

$$\log(\frac{1}{2}) + \log(\frac{n_2}{2n_1}).$$

Since this expression is decreasing in the ratio $n_1/n_2$, the utility of type-1 consumers decreases when the ratio increases. This is the typical situation in which increasing the supply of good 1 relative to good 2 causes the relative price to fall, hurting those who are net suppliers of good 1. It is possible, though, that for different utility functions with strong income effects, the effect could go the other way.