The Inflation Premium Implicit in the US Real and Nominal Term Structures of Interest Rates

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Monthly data updates will be posted at **http://www.econ.ohio-state.edu/jhm/ts/ts.html .**

Abstract

The Inflation Premium implicit in the US Real and Nominal
Term Structures of Interest Rates.

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Monthly term structures are fit to US Treasury inflation-indexed securities using a QN (Quadratic-Natural) spline, developed in this paper, and also to conventional nominal securities of comparable maturities. The ratio of the real to nominal discount functions is an implicit forward CPI function. The difference between the nominal and real forward interest rate curves is an implicit marginal inflation premium. It is demonstrated that under consumption risk-neutrality per Stanley Fischer (1975), this inflation premium does not equal expected future inflation per Irving Fisher (1896,1930), but rather incorporates a weighted average of expectations about the future course of inflation, that tends to give greater weight to low inflation scenarios than to high.

The method is applied to 29 dates since the introduction of the 30-year indexed bond in April 1998. Nominal interest rate volatility is 2-2.5 times greater (in terms of standard deviation) than real interest rate volatility, nominal rate shocks are highly correlated with shocks to the inflation premium, and real interest rate shocks are nearly orthogonal to inflation premium shocks. To date, there is no evidence against the log expectations hypothesis for real interest rates, nor against the Fisher hypothesis for the inflation premium. There is only weak evidence against the Fischer hypothesis.

No evidence is found that the estimated forward rate beyond 30 years is nondecreasing over time, or even has lessened variance, despite the argument of Dybvig, Ingersoll and Ross (1996) that the asymptotic long-term forward rate and zero-coupon rate cannot fall without generating arbitrage opportunities.

Introduction.

The introduction of 30-year price-level indexed US Treasury bonds in April of 1998, in conjunction with 10- and 5- year indexed notes released in January and July of 1997, provides an unprecedented opportunity to measure the US real term structure of interest rates over a broad spectrum of maturities, and to investigate the behavior of both real interest rates and the implicit inflation premium relative to a comparable nominal term structure.[1] These securities are known as Treasury Inflation-Protection Securities or TIPS.

The present paper uses the inflation-indexed issues to obtain a real term structure by means of a Quadratic-Natural (QN) cubic spline functional form, developed here. This functional form is also fit, for comparison, to selected conventional nominal issues. The ratio of the real to nominal discount functions is an implicit forward CPI curve. Under risk-neutrality with respect to consumption uncertainty, as proposed by Stanley Fischer (1975), the reciprocal forward CPI should equal the market's expectation of the future purchasing power of money.

The difference between the nominal and real forward rate curves is an implicit marginal inflation premium. It is demonstrated that under the (Stanley) Fischer hypothesis, the marginal inflation premium reflects not expected future inflation itself per the traditional (Irving) "Fisher Equation," but rather a certain weighted average of conditionally expected future inflation rates, conditioning on the future purchasing power of money.

The method is applied to data on the full term structure for April 1998 through August 2000. This data is used to investigate the real term premium, the Fischer and Fisher hypotheses,

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[1] By mid-2000, there were 7 indexed US Treasury issues outstanding, with a total inflation-adjusted face value of $114.1 billion: an originally 5-year note maturing 7/2002, four originally 10-year notes maturing 1/2007, 1/2008, 1/2009 and 1/2010, and two originally 30-year bonds maturing 4/2028 and 4/2029.

and the behavior of the volatility and correlations of real and nominal forwards and the inflation premium across horizon.

According to Dybvig, Ingersoll and Ross (1996), the asymptotic long-term real or nominal forward rate, if it exists, cannot fall without generating arbitrage opportunities. It is investigated whether the estimated long-term forward rates extracted from this data are consistent with this DIR theorem.

The estimated term structures, forward CPI curves, and inflation premia are archived on the World Wide Web, and will be updated monthly.

Term Structure Notation and Identities:

Let $d(m)$ represent the value at a given point in time of a promise to pay \$1 (or \$1 indexed for inflation) m years in the future. The continuously compounded zero-coupon nominal (or real) yield to maturity is then defined as

$$r(m) = -\log d(m) / m. \tag{1}$$

The instantaneous nominal (or real) forward rate for maturity m is defined as

$$f(m) = -d'(m) / d(m). \tag{2}$$

The above implies

$$r(m) = \frac{1}{m}\int_0^m f(m)dm, \tag{3}$$

so that the zero-coupon yield is an equally-weighted average of forward rates out to the maturity in question. Multiplying (3) through by m and differentiating implies

$$f(m) = r(m) + mr'(m), \tag{4}$$

so that the forward curve is continuous if and only if the zero coupon yield curve is continuously differentiable.

The (continuously compounded) par bond yield for maturity m is defined to be that continuous coupon rate that would just make a bond of maturity m sell at par.[2] This may be computed as:

$$y(m) = \frac{1 - d(m)}{\int_0^m d(m)dm} . \tag{5}$$

Using (2), the above is equivalent to

$$y(m) = \frac{\int_0^m f(m)d(m)dm}{\int_0^m d(m)dm} , \tag{6}$$

so that the par bond yield is also a weighted average of forward rates over the relevant maturity interval, but with weights that decline in proportion to the discount function (cp (3), where the weights are equal instead). This is an exact relationship that does not depend on any approximation.[3]

Multiplying (6) through by the denominator of the right hand side, differentiating and rearranging yields

$$f(m) = y(m) + \frac{\int_0^m d(m)dm}{d(m)} y'(m) , \tag{7}$$

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[2] See McCulloch (1971, 1975) for development of these concepts. The symbols used here differ somewhat from those used in the earlier papers, and instead conform more with current usage. A semiannually compounded interest rate R^s may be computed from a continuously compounded rate R^c by the formula $R^s = 2 (\exp(R^c/2)-1)$, where both are expressed as a fraction of unity rather than as a percentage.

[3] Equation (6) was first developed in McCulloch (1977), but has not been previously published. Shiller (1979) develops a similar, but only approximate, formula that substitutes a constant geometric decay function for the actual discount function. Equation (6) is based on continuous coupons and continuously compounded instantaneous forward rates, but an analogous exact formula obtains with discrete coupons in terms of discretely compounded forward rates.

so that the forward curve is likewise continuous if and only if the par bond yield curve is continuously differentiable.

Joan Robinson once objected to the version of the Expectations Hypothesis that relates forward interest rates to expectations of future short term interest rates, on the grounds that if it were true, anyone who bought an infinite-maturity consol would have "to think he knows exactly what the rates of interest will be every day from now to Kingdom Come" (1951, 102n). However, equation (6) demonstrates that one would not have to have precise expectations about these future rates in order to price consols to any desired precision. In fact, for any given pricing precision, one's ignorance regarding these rates can increase without bound as maturity increases toward Kingdom Come. The obverse of this proposition is that when the forward curve is inferred from actual bond prices by means of (7), it has an intrinsic tendency to become increasingly poorly defined as m becomes very large, unless some a priori structure is imposed upon it, as is done below.

The QN Spline

McCulloch (1975) fits a *cubic spline* to the discount function itself. A cubic spline is a piecewise cubic function that is twice continuously differentiable. The third derivative may be discontinuous at selected points called *knotpoints*. A cubic spline discount function has the advantage of making the pricing equation for coupon bonds linear in unknown parameters that can then be found by least squares. This method is used in McCulloch and Kwon (1993), using data on most of the outstanding Treasury bills, bonds and notes, and estimating far fewer parameters than observations by means of a weighted least squares regression.

However, a spline discount function has the great disadvantage that a cubic often fits the long end of the discount function, where observations are relatively sparse, very poorly (see Vasicek and Fong 1977, Deacon and Derry 1994a, Bliss 1997). Furthermore, there is no sensible way to extrapolate a cubic spline discount function beyond the longest maturity observed, since any cubic must either be constant or diverge to plus or minus infinity. In mid-1986, there were some months when forward rates estimated by this method actually go negative even before the longest maturity observed (see McCulloch and Kwon 1993).

The present paper instead fits a cubic spline to the (negated) *log* discount function

$$j(m) = -\log d(m) \,. \tag{8}$$

In terms of this function,

$$\begin{aligned} r(m) &= j(m)/m \\ f(m) &= j'(m). \end{aligned} \tag{9}$$

This method has the advantage that it can be extrapolated sensibly by a straight line to infinity, simply by imposing the "*natural*" restriction

$$j''(m) = 0 \tag{10}$$

at the longest observed maturity.[4] This in turn implies the discount function is a pure exponential decay beyond this maturity. The forward rate in these maturities is constant, and the zero-coupon yield curve asymptotes to this constant hyperbolically by (3). The par bond yield curve also asymptotes, to a consol interest rate that is in general a different numerical value because of the weighting in (6).

A spline log discount function has the disadvantage that the pricing equation for coupon-bearing bonds, given by (A-3) in the Appendix, becomes a nonlinear function of the unknown parameters, that must be fit by iterative numerical methods. However, with modern computers this is no longer a significant constraint.[5]

There are over 200 primary nominal US Treasury bonds, notes, and bills outstanding, so that an over-identified cubic spline log-discount function with a natural restriction at the long end could easily be fit by least squares or other curve-fitting criterion to this data.[6] However, real term structure data at present has the special limitation that there are only a handful of distinct maturities outstanding. The present paper therefore addresses the special problem of fitting a just-identified term structure to a small number of well-spaced maturities.

Suppose we have n securities with well-spaced maturities m_i, arranged in order by increasing maturity. These maturities, together with $m_0 \equiv 0$, define n+1 knots, and therefore n inter-knot intervals. A completely unconstrained cubic spline with n ntervals will have n+3 parameters – 4 for the cubic in the first interval, and a new third derivative in each of the n-1 subsequent intervals. The condition

$$d(0) \equiv 1, \tag{11}$$

together with the n bond prices and a natural restriction (10) at $m = m_n$, still leave one parameter undetermined. We therefore need one more restriction in order for the n prices to completely determine the curve. A second natural restriction at the short end (i.e. $j''(0) = 0$) is undesirable, because, as can easily be shown from (9), this would imply $r'(0) = 0$ and $f'(0) = 0$, while in fact

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[4] A "natural" spline is a cubic spline whose second derivative is zero at both the first and last data points. A spline with a natural restriction at the long end only could perhaps be referred to as a "seminatural" spline.
[5] Numerous authors (eg Adams and Van Deventer 1994, Fisher, Nychka and Zervos 1995, Waggoner 1997, Evans 1998) have effectively fit the log discount function with various types of spline or exponential decay functions, not always imposing the linear long-maturity behavior mentioned above. Others (Nelson and Siegel 1987, Deacon and Derry 1994a) fit the bond yield curve directly. See Bliss (1997) for a survey of this literature. The present paper is the first to fit a QN spline.
[6] See Appendix 2 below for details.

the term structure ordinarily exhibits pronounced upward slope at maturity 0. We therefore instead impose the *quadratic* restriction

$$j'''(m) = 0, \quad m \in [= 0, m_1] \tag{12}$$

throughout the first interval. This restriction allows the yield and forward curves to have any slope at m = 0, and instead merely constrains them to be linear out to the first knot point. We refer to the resulting curve as a *Quadratic-Natural cubic spline*, or *QN spline* for short.

We then use an iterative procedure to find the unique QN spline log discount function that explains the observed bid-asked mean prices of the securities used, in terms of the present discounted value of their coupon and principal payments. Appendix 2 gives details of the computation.

The *Wall Street Journal*, the Bloomberg Screen and Website[7] and, most recently, the Federal Reserve Bank of St. Louis *Monetary Trends* depict yield curves that exactly fit the yields to maturity of selected benchmark issues. As long as the selected issues are selling at or very near par, these represent valid points on the par bond yield curve. However, these sources all depict the yield curve elsewhere as piecewise linear. Equation (7) above demonstrates that a piecewise linear par bond yield curve, with a discontinuous derivative, implies a *dis*continuous forward curve, even if the par bond yield curve itself is continuous. A discontinuous forward curve in turn implies either implausible expectations about future short-term interest rates or implausible expectations about holding period returns. The present approach instead fits, in effect, a smooth par-bond yield curve to the observed yields, consistent with a forward curve that is not just

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[7] At screen C13 on the screen service and at http://www.bloomberg.com/markets/C13.html on the web.

continuous, but continuously differentiable. It therefore renders piecewise linear yield curves obsolete.[8]

The Real Term Structure

Figure 1 shows the Real US Treasury Term Structure for August 31, 2000, using *Wall St. Journal* quotes on the indexed notes maturing in 7/2002, 1/2007,1/2010, and the indexed bond maturing in 4/2029. The securities maturing in 1/2008, 1/2009 and 4/2028, being proportionally quite close to these, could easily cause wild gyrations in the forward curve if included with an exact fit method within the limits defined by transactions costs, and so were excluded.

[8] The Bloomberg screen service does have a continuous quadratic forward curve option available for its swaps term structure (screen FWCV US, with fitting option 2 selected on screen SWYC). However, this is apparently not yet available for the C13 on-the-run Treasury term structure.

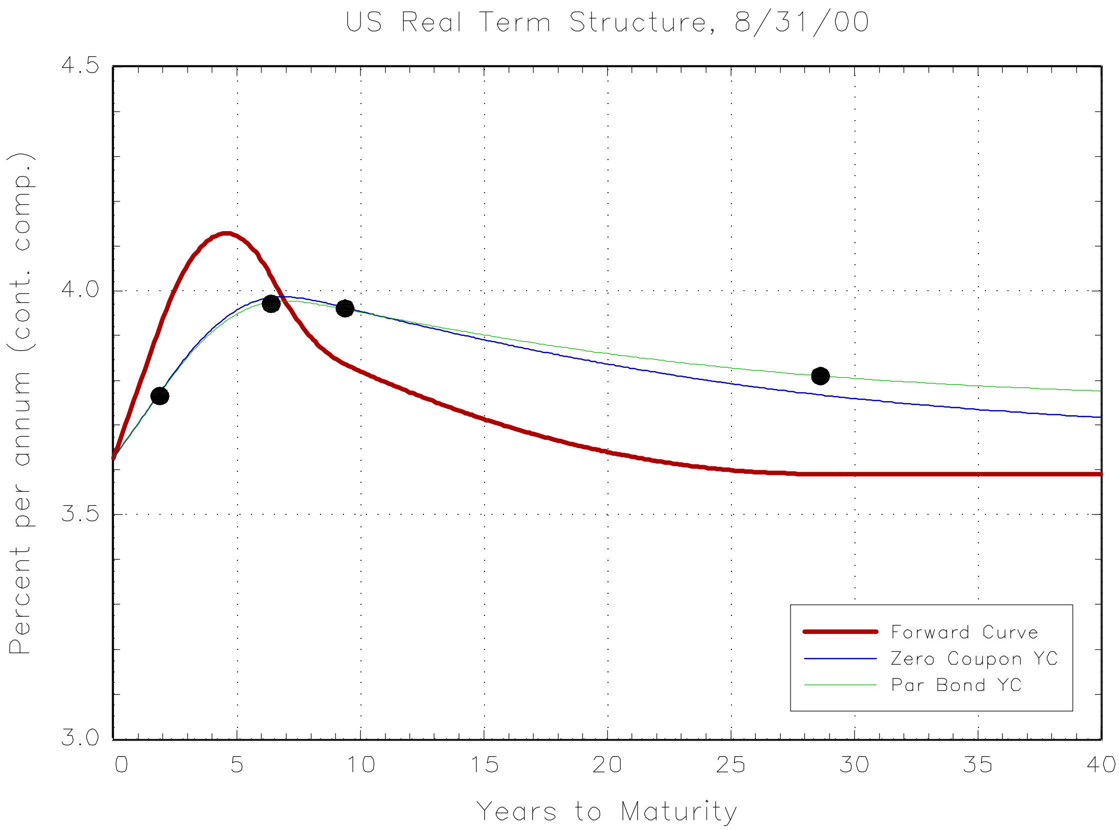

Figure 1

US Real Term Structure for 8/31/2000.

The dots in Figure 1 indicate the continuously compounded[9] asked yields to maturity of the securities used, plotted relative to their actual remaining maturities. The estimation is based on the quoted bid/asked mean prices rather than these yields, but the dots should lie on or nearly on the estimated par bond yield curve if the issues in question indeed sell near par and the bid/asked spread is sufficiently small.

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[9] The bond equivalent, or semiannually compounded. yields conventionally reported on securities will lie slightly above these continuously compounded par bond yields. (See footnote 2 above.) However, semiannual compounding unnecessarily complicates the computation and interpretation of an implicit inflation premium.

The QN spline allows all three curves in Figure 1 to be extrapolated sensibly as far as desired (to 40 years in these figures). The real forward curve is, by assumption, flat beyond the longest observed maturity of 28.38 years, at an estimated level of 3.59%. The zero coupon curve eventually asymptotes to this value by equation (3). The par bond yield curve is a dampened version of the zero-coupon curve. It asymptotes much more quickly than does the zero coupon curve, to a hypothetical real consol interest rate that may be numerically different from the limiting forward and zero coupon rate, in this case 3.73%.

Other, slightly differently shaped term structures will also fit these data points exactly. However, these will have to have approximately the same average forward rates between observations, and so cannot differ by much if they are equally smooth. If a new 20- or 40-year indexed Treasury bond were issued at par, its yield would of course not lie exactly on the respectively interpolated or extrapolated portions of the par bond yield curve in Figure 1. Nevertheless, this curve gives an educated estimate of where the yields on such new bonds would lie.[10] Corporate indexed bonds would be expected to lie above this US Treasury curve according to their tax status and respective ratings.

Beginning in April 2000, four TIPS were used to fit the real term structure, as in Figure 1. Between April 1998 and March 2000, only three TIPS were used, specifically the most recently issued (and therefore longest maturity) security in each of the three available maturity sectors. Between July 1997 and March 1997, there were only 5- and 10-year indexed notes. For the purpose of the illustrations below, a term structure was fit with only these two maturities. However, since the data contained no real information on the term structure beyond 10 years,

[10] McCulloch (1993) argues that it is realistic to view the state of expectations about future output, therefore about the term structure, as infinite dimensional. This is nevertheless consistent with the explanatory power of additional parameters declining rapidly as the number used increases.

these months were not included in the drift, variance and covariance calculations below. Between January 1997 and June 1997, only one indexed note, maturing 1/2007, was outstanding. With only one observation, a QN spline log discount function reduces to a straight line through the origin, implying a flat forward and yield curve. This rate was included in the plots below to give a feel for the early history of US real rates, but not in the aforementioned calculations.

The Nominal Term Structure

There are over 200 outstanding primary nominal US Treasury securities whose prices are quoted daily in the financial press. These define a nominal term structure with far more shape than is possible with the indexed issues, even when callable and inactive issues are excluded. Comparing the real term structure of Figure 1 to such a nominal term structure would result in an implicit inflation premium with considerable spurious shape, particularly in the first two years, that was coming entirely from the nominal side. In order to obtain a nominal term structure that has no more and no less shape than the real term structure, it is therefore expedient to compute a nominal term structure, for purposes of comparison to the real term structure only, that is based on nominal securities of exactly the same number and as nearly as possible the same maturities, as the real term structure.

One's first choice would be to use the most recently issued, and therefore most liquid, "on-the-run" securities for this nominal comparison term structure. However, these often sell at a substantially lower yields than off-the-run securities of comparable maturities, apparently due to their desirability as collateral for repurchase agreements (see Duffie, 1996). These low yields make them relatively unattractive as pure investments on the basis of their observable, promised payments alone, so that using them would make the inferred inflation premium unrealistically small.[11] Accordingly, our comparison nominal term structures are based on the highest yield reasonably liquid (in terms of small bid/asked spread) non-callable securities that can be found with approximately the same maturities as the indexed bonds employed. Ordinarily these will be off-the-run issues. Figure 2 below, for example, is based on the 6s of 7/2002, the 6.25s of 2/2007, the 6s of 8/2009, and the 6.125s of 11/2027.

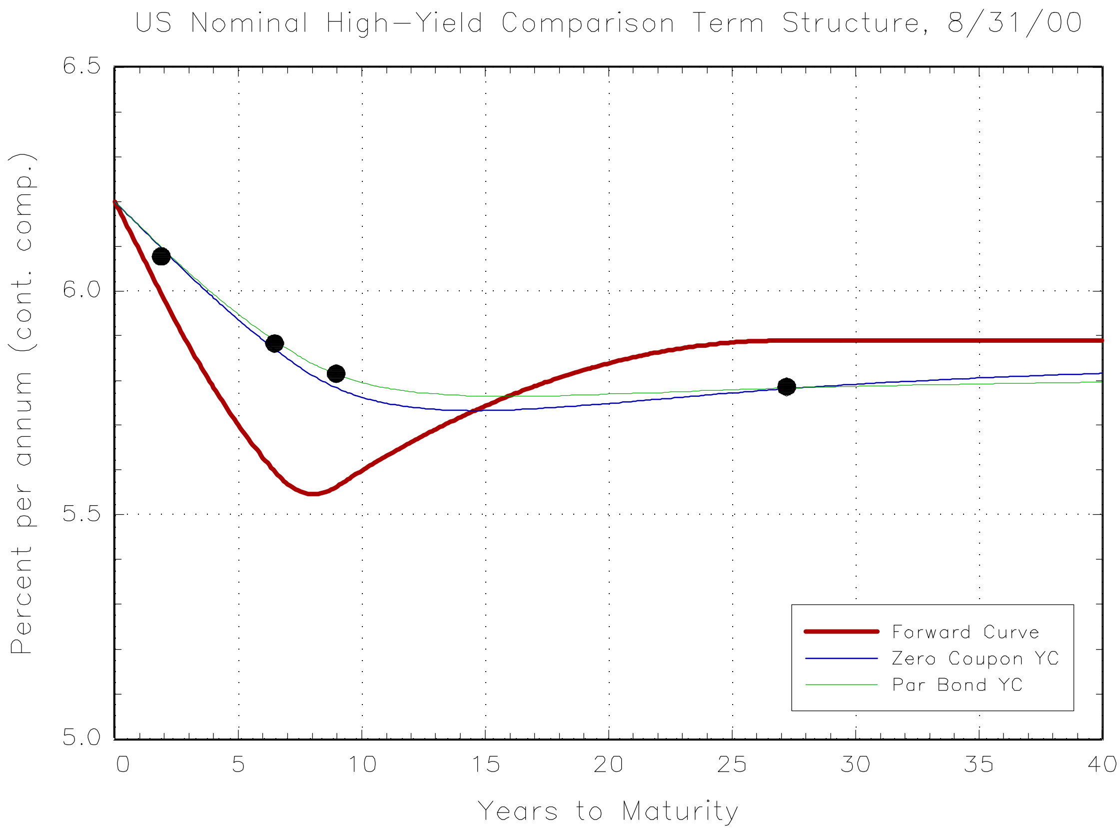

Figure 2

Comparison nominal high-yield Term Structure for 8/31/2000.

[11] On 2/17/00, for example, investors who bought the new bond maturing in 5/2030 in preference to the "pre-owned" 2/2029 were actually paying the Treasury 3% per annum to keep their principal for an extra 1.25 years. Similarly, those who bought the 2/2029 in preference to the musty 11/2027 received essentially nothing for letting the Treasury keep their principal for that 1.25 year interval. See McCulloch (2000).

The forward curve is again flat, by assumption, beyond approximately 27 years, at an estimated level of 5.89%. The nominal zero coupon yield curve eventually asymptotes to the same value by (3). The par bond yield curve asymptotes to a hypothetical nominal consol interest rate estimated to be 5.80%. The data for Figure 2 and prior months, is likewise archived at http://www.econ.ohio-state.edu/jhm/ts/nom.html .

The Forward CPI and Inflation Premium

Let P_t be the reference CPI for time t, $d^R(t,T)$ be the real discount function at time t for repayment at future date T = t+m, and $d^N(t,T)$ be the nominal discount function at time t for repayment at future date T. Then $P^F(t,T) = P_t d^R(t,T)/d^N(t,T)$ is an implicit forward reference CPI at time t for future date T. This is plotted for the data of Figures 1 and 2 in Figure 3, shifted left by 2.5 months to allow for the lag in indexation.[12]

[12] The indexation lag is discussed at greater length in Appendix 1 below.

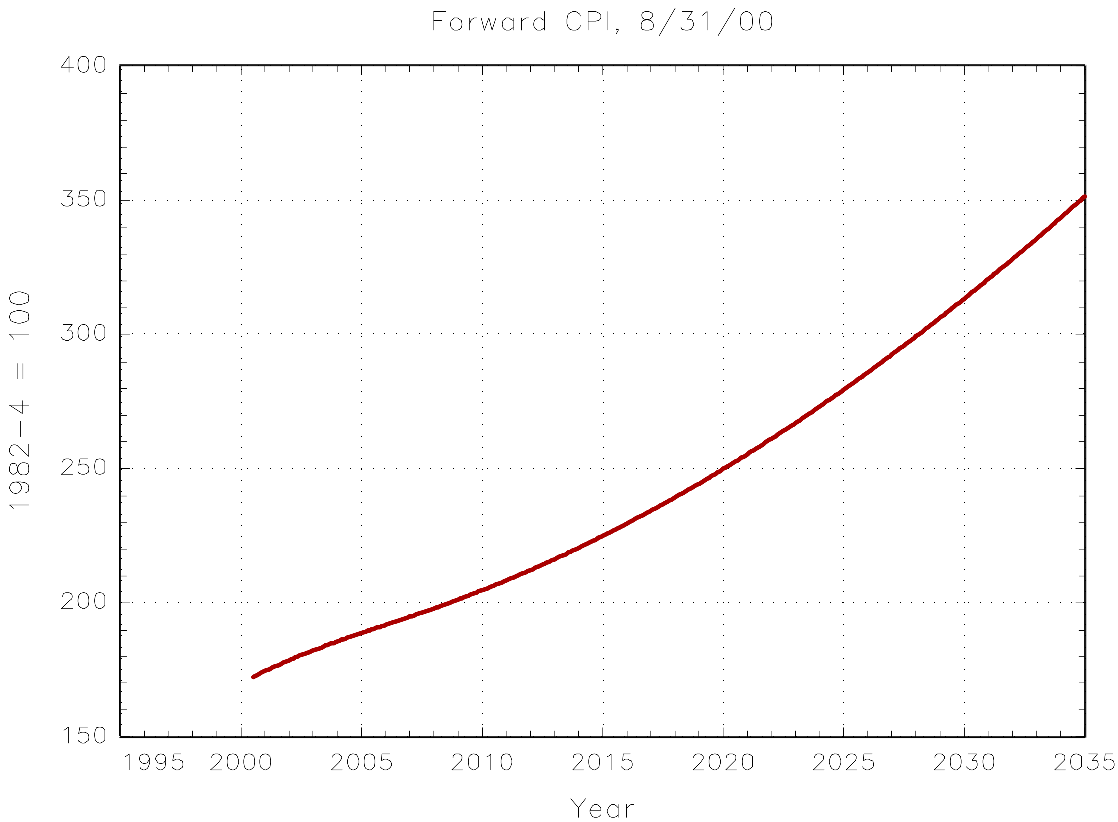

Figure 3

The forward CPI for 8/31/00.

Let $f^R(t,T)$ $(f^N(t,T))$ represent the real (nominal) forward rate in the market at time t for future date T = t+m. Then the instantaneous rate of growth of the forward CPI, or equivalently the nominal forward curve minus the real forward curve,

$$\begin{aligned} \pi^F(t,T) &= \partial \ln P^F(t,T)/\partial T \\ &\equiv f^N(t,T) - f^R(t,T), \end{aligned} \qquad (13)$$

is the implicit marginal inflation premium for future date T. This is shown as the heavy line in Figure 4 for the data of Figures 1 and 2, again shifted 2.5 months to allow for the indexation lag.[13] The marginal inflation premium is, by construction, a constant, here estimated to be 2.30%, beyond the longest observed real or nominal bond maturity.

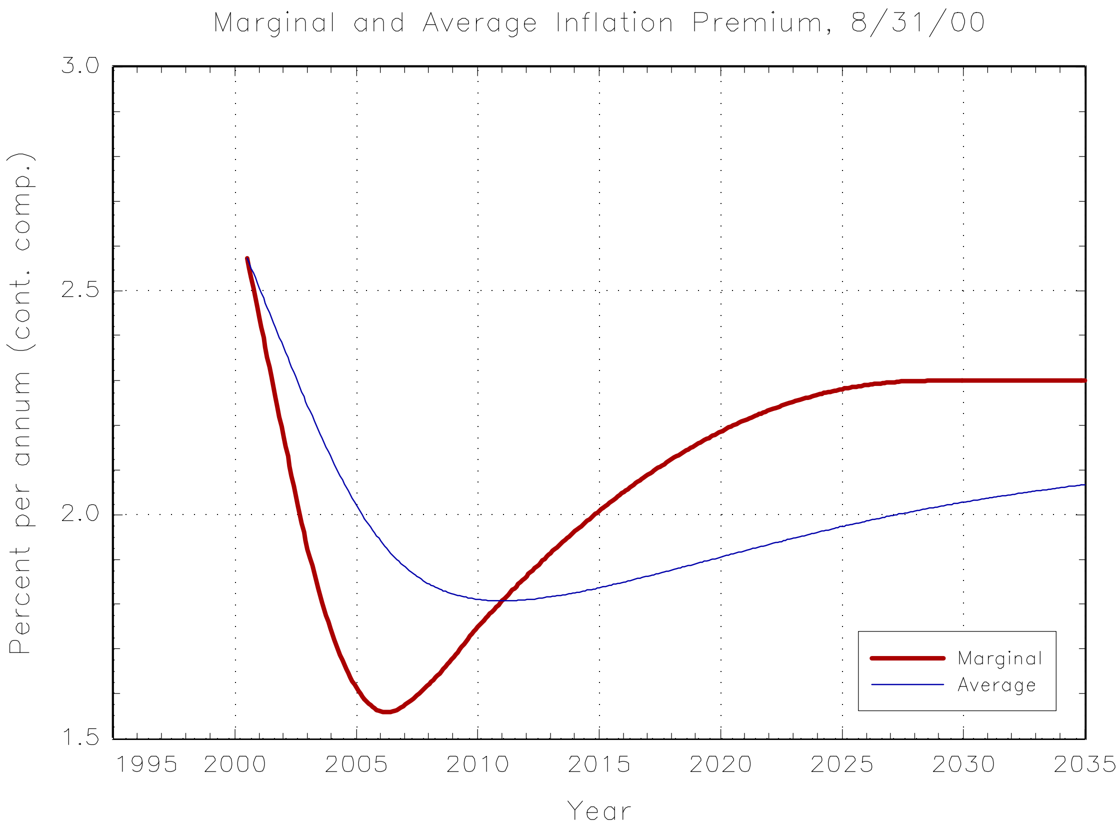

Figure 4

The marginal and average inflation premium for 8/31/00.

The thin line in Figure 4 gives the average inflation premium,

[13] The difference between the two zero-coupon yield curves gives an implicit *average* inflation premium, bearing the same relation to the marginal inflation premium that an average cost or revenue schedule bears to a marginal cost or revenue schedule. This is not depicted here, but may be computed from the web-archived data as desired.

$$p^A(t,T) = \ln P^F(t,T)/m \\ \equiv r^N(t,T) - r^R(t,T), \quad (14)$$

where $r^R(t,T)$ etc. is the real zero-coupon yield at time t for repayment at future date T = t+m. As with forward and zero-coupon yields in equation (4),

$$p^F(t,T) = p^A(t,T) + (T-t)(\partial p^A(t,T)/\partial T),$$

so that $p^F(t,0) = p^A(t,0)$ (so long as $\partial p^A(t,m)/\partial m$ is finite), $p^F(t,m) > (=,<) p^A(t,m)$ as $\partial p^A(t,m)/\partial m > (=,<) 0$, and whenever the marginal inflation premium has an asymptotic value, the average premium has the same asymptotic value.

The Inflation Premium and Uncertainty about the Future Price Level: Fisher vs. Fischer[14]

In a world of complete markets, zero transactions costs, and perfect foresight the forward CPI must equal the actual future CPI for the date in question, after adjustment for the indexation lag. However, with uncertainty and risk neutrality with respect to consumption risk, we would not expect the forward CPI to equal the expected future CPI, nor even would we expect its logarithm to equal the expected future log CPI. Rather, as noted by Stanley Fischer (1975, 518n), risk neutrality should equate the contractual real return to maturity on indexed bonds to the expected real return to maturity on nominal bonds. Ignoring, for the moment, the lag in indexation, this may be written:

$$\frac{1}{d^R(t,T)} = E_t \frac{P_t}{P_T d^N(t,T)} \quad (15)$$

This "[Stanley] Fischer equation" implies the *reciprocal* forward CPI should equal the expected future *reciprocal* CPI:

$$Q^F(t,T) = E_t Q_T \, , \tag{16}$$

where $Q_t = 1/P_t$ is the purchasing power of money at time t and $Q^F(t,T) = 1/P^F(t,T)$ is the forward purchasing power of money at time t for future date T = t+m. By Jensen's Inequality, the three values are not equal:

$$E_t P_T > \exp E_t \ln P_T > 1/E_t(1/P_T) = P^F(t,T) \tag{17}$$

The differences can be quite large at long horizons where the uncertainty is greatest:[15]

It may be shown that when consumption risk-neutrality prevails as in (16) and (17), the marginal inflation premium implicit in the forward CPI curve will not equal expected future inflation per the famous (Irving) Fisher equation (1896, 1930), nor even the negative of the expected future appreciation rate of money, but rather the negative of a *weighted average* of conditionally expected future appreciation rates, each conditional on the future purchasing power of money, and weighted both by the probability of the future purchasing power and by the purchasing power itself.

To see this, let $p(Q_T)$ be the probability density (based on time t information) of Q_T etc. Then under purchasing power risk-neutrality, the (Stanley) Fischer equation (16) implies

$$\begin{aligned} Q^F(t,T) &= E_t Q_T \\ &= \int_0^\infty Q_T \, p(Q_T) dQ_T \end{aligned} \tag{18}$$

and

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[14] The present section draws heavily on Kochin (1980). The remainder of the paper is primarily due to McCulloch.

[15] As noted by McCulloch (1996, 410-11), if the log CPI is *stable* with characteristic exponent $\alpha < 2$, it is natural to assume that its distribution is maximally positively skewed, ie skewness parameter $\beta = 1$, simply because rapid inflation is fiscally much more attractive than comparably rapid deflation. In this case the reciprocal CPI will have finite expectation, while the CPI itself will actually have infinite mathematical expectation. Bidarkota and McCulloch (1998) demonstrate that CPI inflation exhibits significant leptokurtosis, and assuming stability find an

$$
\begin{aligned}
Q^F(t,T+dt) &= E_t Q_{T+dt} \\
&= \int_0^\infty Q_{T+dt}\, p(Q_{T+dt})\, dQ_{T+dt} \\
&= \int_0^\infty Q_{T+dt} \int_0^\infty p(Q_T, Q_{T+dt})\, dQ_T\, dQ_{T+dt} \\
&= \int_0^\infty Q_{T+dt} \int_0^\infty p(Q_T)\, p(Q_{T+dt} \mid Q_T)\, dQ_T\, dQ_{T+dt} \qquad (19) \\
&= \int_0^\infty Q_T\, p(Q_T) \int_0^\infty \frac{Q_{T+dt}}{Q_T}\, p(Q_{T+dt} \mid Q_T)\, dQ_{T+dt}\, dQ_T \\
&= \int_0^\infty Q_T\, p(Q_T)\, E_t\left(\frac{Q_{T+dt}}{Q_T} \mid Q_T \right) dQ_T
\end{aligned}
$$

It follows that

$$
\exp\left(-p^F(t,T)dt\right) = \frac{Q^F_{T+dt}}{Q^F_T} = \frac{\int_0^\infty Q_T\, p(Q_T)\, E_t\left(\frac{Q_{T+dt}}{Q_T} \mid Q_T \right) dQ_T}{\int_0^\infty Q_T\, p(Q_T)\, dQ_T}, \qquad (20)
$$

so that

$$
p^F(t,T) = \frac{\int_0^\infty Q_T\, p(Q_T)\, E_t\left(\frac{-Q'_T}{Q_T} \mid Q_T \right) dQ_T}{\int_0^\infty Q_T\, p(Q_T)\, dQ_T}, \qquad (21)
$$

as claimed.

In particular since, under modern fiat money conditions, future inflation is undoubtedly positively correlated with the future level of the CPI, and hence the future appreciation rate Q_{T+dt}/Q_T is positively correlated with the future purchasing power of money, the forward inflation rate at distant horizons must give more weight to low inflation scenarios than to high inflation scenarios: If 30 years from now the price level is very high relative to today's expectations, inflation is also likely to be very high, but by then the value of money will have been virtually extinguished, so that these high inflation rates will make little difference for the present value of

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estimated α value of 1.83, although for reasons of computational expediency they restrict themselves to the nominal bonds. Those scenarios in which future prices turn out to be relatively low, and in which inflation is therefore likely to continue to be relatively low, have a disproportionately large impact on the pricing of nominal bonds today, relative to their probabilities.

Klein noted already in 1975 that since World War II, the U.S. inflation rate had been behaving very nearly like a random walk, or an I(1) process in today's parlance, rather than having clear mean-reverting tendencies as previously (Klein 1975, Stec 2000). If the inflation rate were truly a pure random walk, the positive correlation between future inflation and the future price level would be particularly strong. The variance of the inflation rate would increase in proportion to maturity, while the variance of the log price level itself would increase with the *cube* of maturity. The marginal inflation premium would then actually be a concave quadratic in maturity, despite the fact that expected future inflation would be constant at its most recent level.[16] In practice, the inflation process is likely to be asymmetrical with respect to inflation and deflation, so that a pure random model should not be taken literally, but it does demonstrate that the conditional inflation effect can be quite powerful.

Since 1984, some evidence for mean reversion in postwar US inflation has been accumulating.[17] Nevertheless, even if high inflation invariably eventually comes back down, the evidence remains that any future upturn in inflation may linger for decades, still creating a strong and positive correlation between the future price level and future inflation.[18]

In addition to the strong Jensen's Inequality effect noted above, there may or may not also be a weaker Jensen's Inequality effect separating conditionally expected future inflation from the

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symmetric ($\beta = 0$) stable case.

[16] See Kochin (1980) for details.

[17] See Stec (2000), Figure 3.2.

negative of the conditionally expected future appreciation rate. To see this, assume that future inflation p_T equals the sum of a stochastic inflation trend $\bar{p}_T$, plus white noise:

$$\begin{aligned} d\log P_T &= \log P_{T+dt} - \log P_T \\ &= p_T dt \\ &= \bar{p}_T dt + t dz, \end{aligned} \tag{22}$$

where dz has variance (dt)[1/2].[19] Assume (again somewhat unrealistically, but for the sake of illustration) that, relative to time t information, and conditional on P_T (and therefore Q_T), $\bar{p}_T$ has a Gaussian distribution with mean $m_{T|P_T}$ and variance σ^2. Then

$$\mathrm{var}_t(p_T dt \mid P_T) = s^2 dt^2 + t^2 dt, \tag{23}$$

whence

$$\begin{aligned} E_t\left(\frac{Q_{T+dt}}{Q_T} \mid Q_T\right) &= E_t\left(\exp(-p_T dt) \mid P_T\right) \\ &= \exp\left(-m_{T|P_T} dt + \frac{1}{2}(s^2 dt^2 + t^2 dt)\right) \\ &= 1 - m_{T|P_T} dt + \frac{1}{2} t^2 dt + O(dt^2). \end{aligned} \tag{24}$$

It follows from (19) that

$$\begin{aligned} p^F(t,T) &= -\log\left(\frac{Q^F_{T+dt}}{Q^F_T}\right) / dt \\ &= \left(1 - \frac{Q^F_{T+dt}}{Q^F_T} + O(dt^2)\right) / dt \\ &= \frac{\int_0^\infty m_{T|P_T} Q_T p(Q_T) dQ_T}{\int_0^\infty Q_T p(Q_T) dQ_T} - \frac{1}{2} t^2. \end{aligned} \tag{25}$$

---

18 During the 19th century, the US price level itself behaved much like a stationary or I(0) series. In such a situation, the correlation between the price level and inflation may be negative. But this has little relevance to 20th century experience, and presumably equally little to the 21st century.

19 Bidarkota and McCulloch (1998) find that, to at least a first approximation, US CPI inflation has behaved like a random walk plus such noise, but with infinite-variance stable errors.

Hence, if future inflation is not noisy about its stochastic trend, ie if $\tau^2 = 0$, then the forward inflation premium just equals the purchasing-power-weighted average of future conditionally expected inflation rates $m_{T|P_T}$. But if future inflation is noisy about its stochastic trend, ie if $\tau^2 > 0$, the forward inflation premium is *even less* than this weighted average. Interestingly, the uncertainty of the future conditional trend inflation itself, σ^2, drops out and contributes nothing to this second Jensen's Inequality effect.

In the real world of risk aversion and imperfect markets, the marginal inflation premium may further contain

- a (positive) risk premium compensating lenders for the risk of high inflation,
- a (negative) risk premium compensating borrowers for the risk of low inflation,
- a (negative) liquidity premium forgone by holders of the more liquid nominal securities, and
- differential tax effects.

Quantification of these factors goes beyond the scope of the present paper.[20]

Empirical Behavior of Real Rates and the Inflation Premium

Figure 5 below shows the real instantaneous forward rate $f^R(t,T)$ for three illustrative future dates T, versus market time t, monthly for the full term structure from 4/98 to 8/00. In order to carry the plot back to the first issue of TIPS in 1/97, the real term structure was fit between 7/97 and 3/98 using only the 5- and 10-year notes. During this period, the rates for 2010 and 2030 coincide as a consequence of the “natural” restriction at the long end of the QN spline. Between 1/97 and 6/97 the real term structure was taken as flat at the yield on the sole 10-year note outstanding, and all three curves coincide. The data prior to 4/98 is not included in the computations reported in this section.

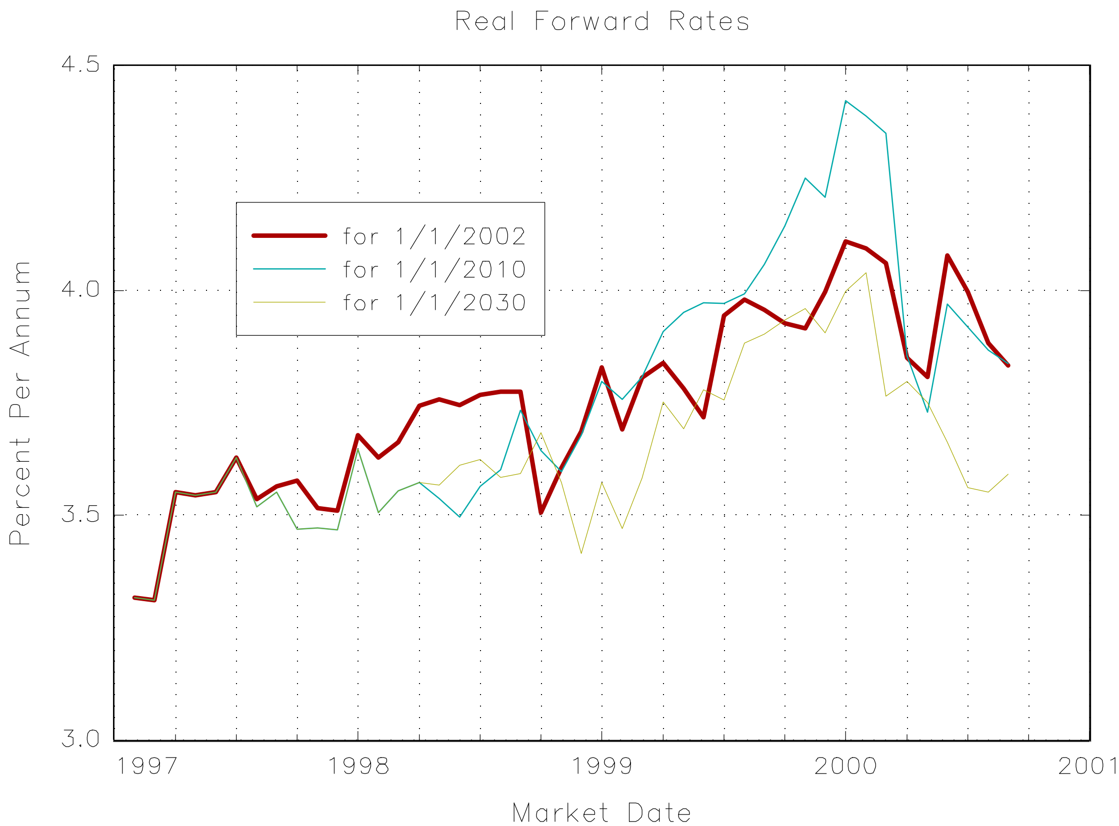

Figure 5

Real instantaneous forward rates for three illustrative future dates.

The real term premium for maturity m may be defined as the excess of the time t real forward rate for future date T = t+m over the time t expectation of the date T 0-maturity real interest rate:

$$TP^R(m) = f^R(t,T) - E_t f^R(T,T) \tag{26}$$

---

[20] Breedon and Chadha (1997) find that the inflation premium in U.K. indexed bonds has generally underpredicted inflation. This is consistent with a positive risk premium compensating lenders for potentially high

Invoking the Law of Iterated Expectations, the time t shift in the real forward rate at horizon m over the observation interval Δt (here 1/12 year) may be written

$$\begin{aligned}\Delta f^R(t,t+m) &= f^R(t+\Delta t,t+m) - f^R(t,t+m) \\ &= (E_{t+\Delta t} f^R(t+m,t+m) + TP^R(m-\Delta m)) - (E_t f^R(t+m,t+m) + TP^R(m)) \\ &= TP^R(m-\Delta t) - TP^R(m) + E_{t+\Delta t} f^R(t+m,t+m) - E_t E_{t+\Delta t} f^R(t+m,t+m) \\ &= TP^R(m-\Delta t) - TP^R(m) + e^R(t,m)\end{aligned} \qquad (27)$$

where $e^R(t,m)$ is a forecasting error whose variance may depend on m. It will be highly correlated across m, but serially uncorrelated across observations. In the absence of a real term premium, i.e. when the benchmark "Log Expectations Hypothesis" holds for the real term structure, $f^R(t,T)$ will therefore be a separate martingale for each T across market time t. If, instead, there is a positive term premium or "liquidity premium," so that $TP^R(m)$ is a monotonically increasing function, $f^R(t,T)$ will shift downward in expectation as t approaches T. Alternatively, if there is a negative real term premium or "solidity premium," $f^R(t,T)$ will shift upward in expectation instead. The sign of the term premium may be shown to depend on whether the volatility of the innovations to future output is an increasing or decreasing function of the horizon m. A zero term premium is a theoretical possibility and a valid benchmark, though not a theoretical necessity.[21]

Using the 29 monthly observations on the full real term structure from 4/98 to 8/00, no statistically significant departure from martingale behavior was found at any maturity from 0 to 480 months. While this is consistent with a 0 term premium at all maturities, it is not conclusive

inflation, although their time series is too short to demonstrate that this underprediction is significant.

[21] See McCulloch (1993), who demonstrates that the Expectations Hypothesis may hold in a continuous time rational expectations equilibrium in terms of continuously compounded interest rates, contrary to the claim to the contrary by Cox, Ingersoll and Ross (1981). For simplicity, we assume here, as a first approximation, that the term premium is time-invariant.

in view of the small sample size. Perhaps as the sample size enlarges, a real term premium will emerge in the US data.

Figure 6 below shows the behavior of the corresponding comparison nominal instantaneous forward rates $f^N(t,T)$ across market time t for the same three illustrative future dates T. A nominal term premium may be defined as in (26), and estimated via the shifts in the nominal forward curve as with (27). Although there is well known to be a positive term premium in the first 6 months or so of the US Treasury nominal term structure (McCulloch 1975, 1987), no significant departure from martingale behavior was found using this data. However, this is not entirely unexpected given the absence of any securities under 2 years to maturity in this data set, and the small sample size.

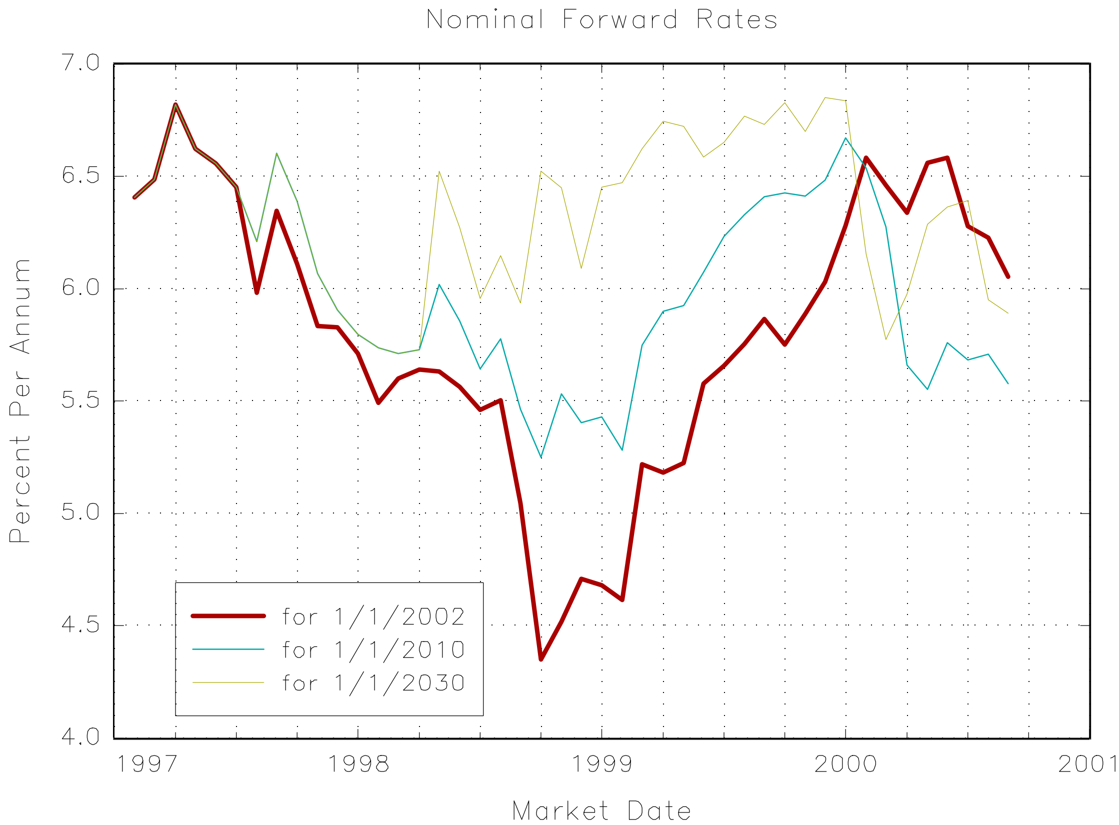

Figure 6.

Comparison nominal instantaneous forward rates for three illustrative future dates.

Figure 7 below shows the marginal inflation premium $p^f(t,T)$ for the same three illustrative future dates, computed simply as Figure 6 minus Figure 5. Under the (Irving) Fisher hypothesis that the inflation premium is an unbiased forecast of future inflation, $p^f(t,T)$ should be a martingale for each T, with perhaps a maturity-specific volatility. Under the (Stanley) Fischer hypothesis that the reciprocal forward CPI is an unbiased forecast of the future reciprocal CPI, on the other hand, the marginal inflation premium should be a downward biased forecast of future inflation if future inflation is positively correlated with the future price level. That is to say, each curve should tend to drift upwards on average over time. Again, no significant drift at any maturity was detected to date, either up or down, using this data set.

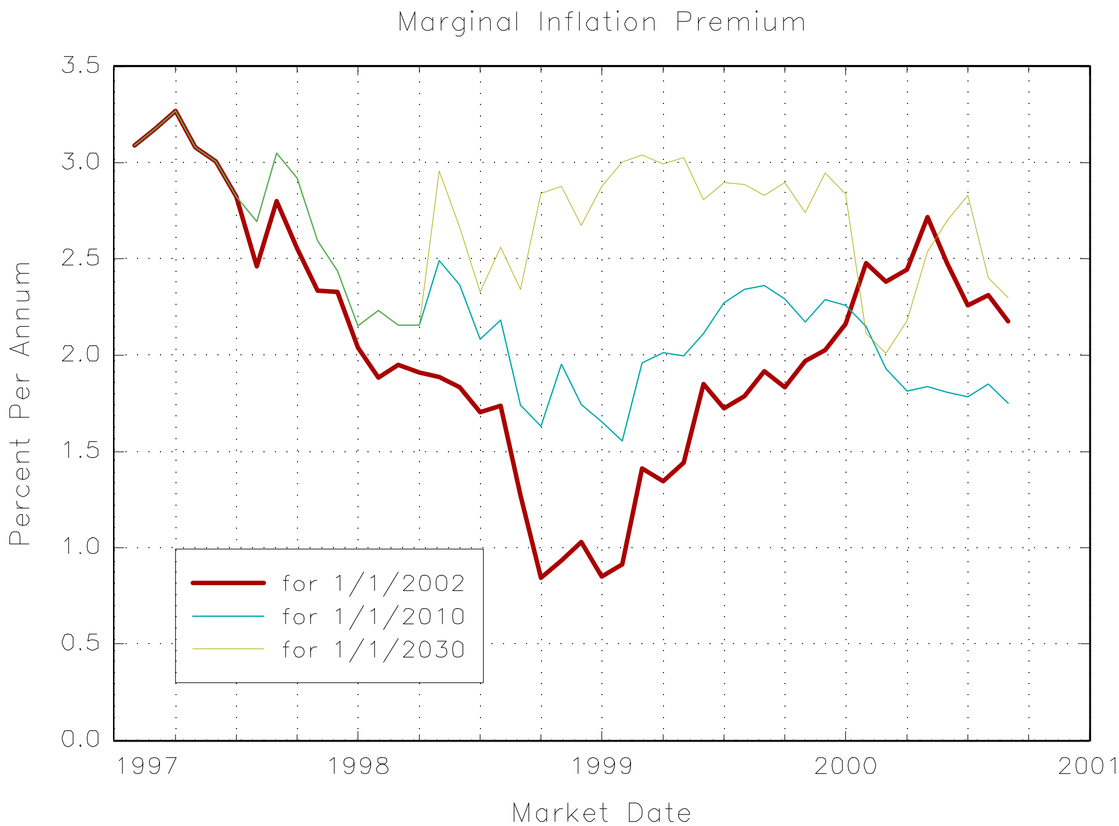

Figure 7.

Marginal inflation premium for three illustrative future dates.

Figure 8 below shows the forward CPI $P^F(t,T)$ for the same three illustrative future dates. The volatility of the forward CPI naturally increases with horizon. Between 4/98 and 8/00, the forward CPI for 1/1/2030 has fluctuated between 300 and 360, the latter being in excess of twice the current level of the CPI.

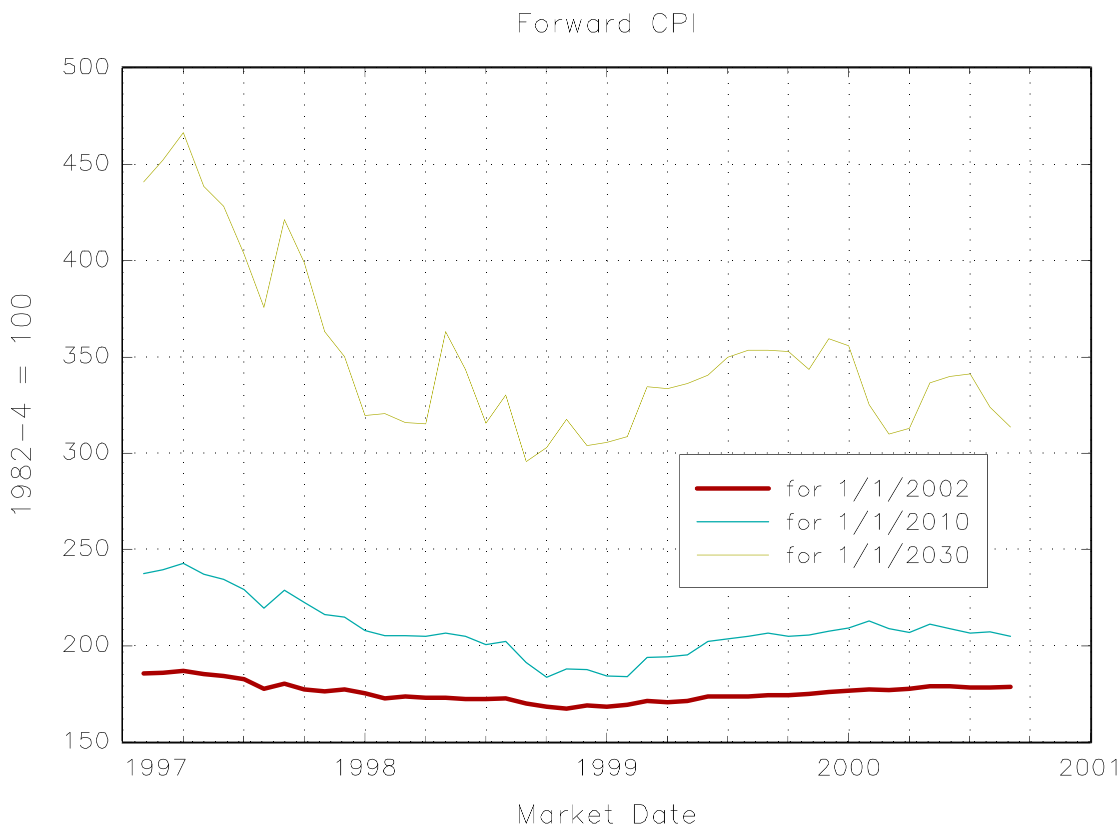

Figure 8.

Forward CPI for three illustrative future dates.

Figure 9 below shows the reciprocal forward CPI $Q^F(t,T)$ for the same three illustrative dates, based to 1982-4 = 100. Under the (S.) Fischer hypothesis, each such curve should trace a martingale over time, regardless of the correlation of the future price level and inflation. Although there was a small downdrift that was just barely significantly negative at the 95% confidence level between maturities 5-10 months, inclusive, this drift was restricted to a maturity range that was entirely dependent on extrapolation of the real and nominal yield curves. While technically this is a departure from the Fischer hypothesis, it remains to be seen if it will hold up with a longer time series of observations.

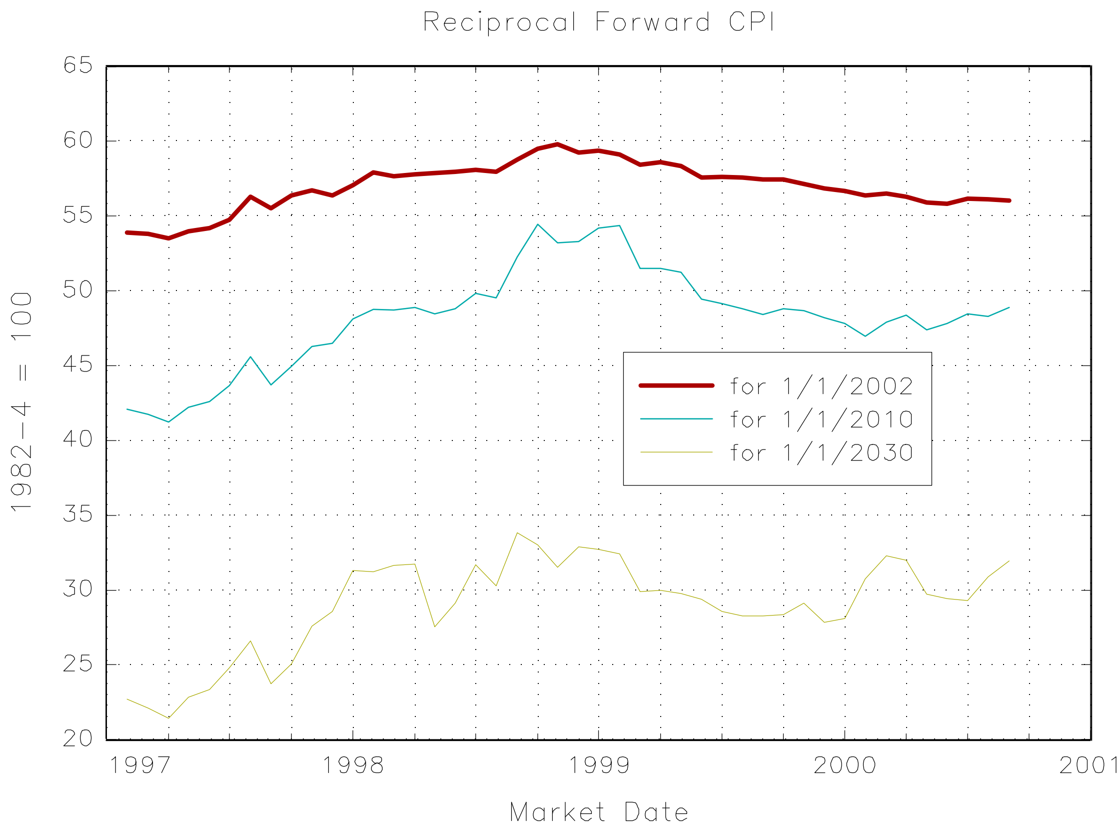

Figure 9.

Reciprocal forward CPI for three illustrative future dates.

Figure 10 below shows the estimated maturity-dependent standard deviations of the shifts in the real forward curve, nominal forward curve, and marginal inflation premium about their estimated means, for 1 through 480 months to maturity. Figure 11 shows the ratio of the nominal to real standard deviation from Figure 10. It may be seen that at most horizons, nominal forward rates are about 2 to 2.5 times more volatile, in terms of standard deviation, than are real forward rates, i.e. about 4 to 6 times more volatile in terms of variance. The volatility of the inflation premium alone is almost as large as that of the nominal rates. (By construction the standard deviations are constant beyond 30 years, since the estimated real and nominal forward rate curves are both flat at these maturities.)

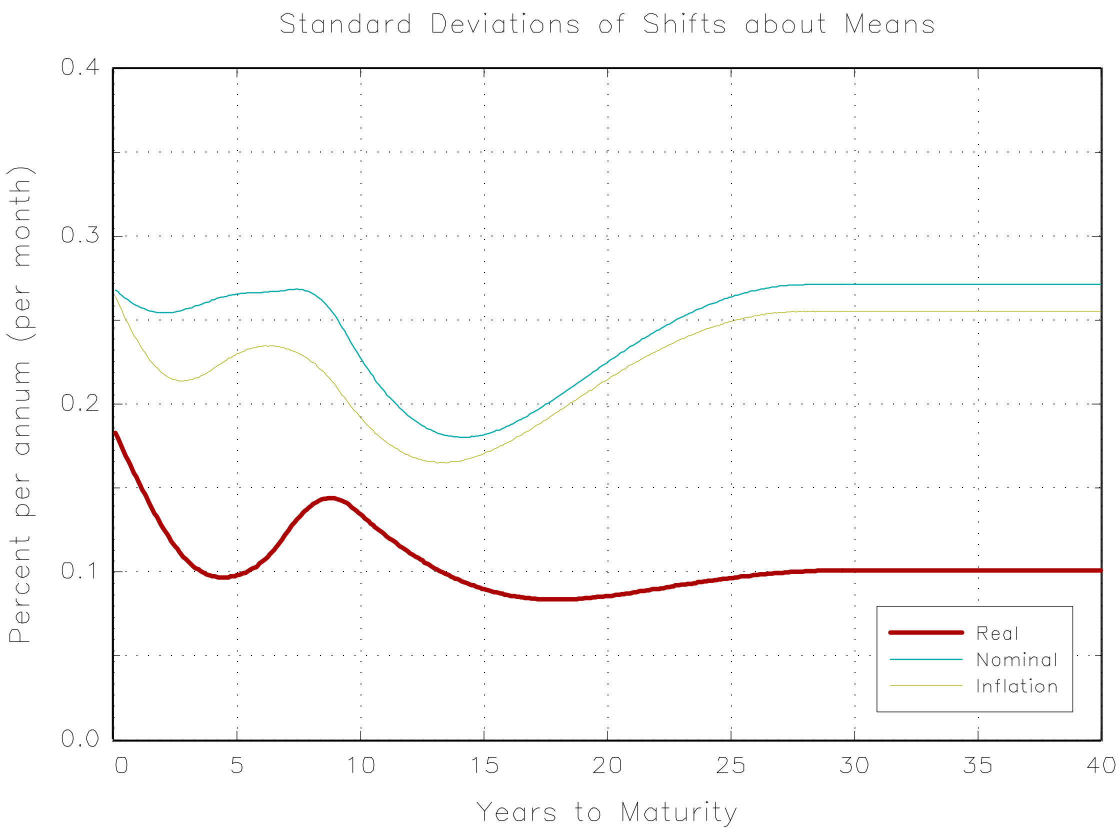

Figure 10.

Standard errors of shifts in real and nominal forward rates, and in the inflation premium, about their estimated means.

Figure 11.

Ratio of standard deviation of nominal forward rate shifts to that of real forward rate shifts.

Finally, Figure 12 below shows maturity-specific estimated correlations among the real and nominal forward rate and marginal inflation premium shifts. It may be seen that at most horizons, the correlation between nominal rates and the inflation premium is between .8 and .9, which corresponds to an R^2 between .64 and .81. The correlation between real rates and the inflation premium fluctuates about 0, and is insignificant at the 95% level at all maturities. Real and nominal rates are positively correlated, with a correlation coefficient between .3 and .5 at most horizons.

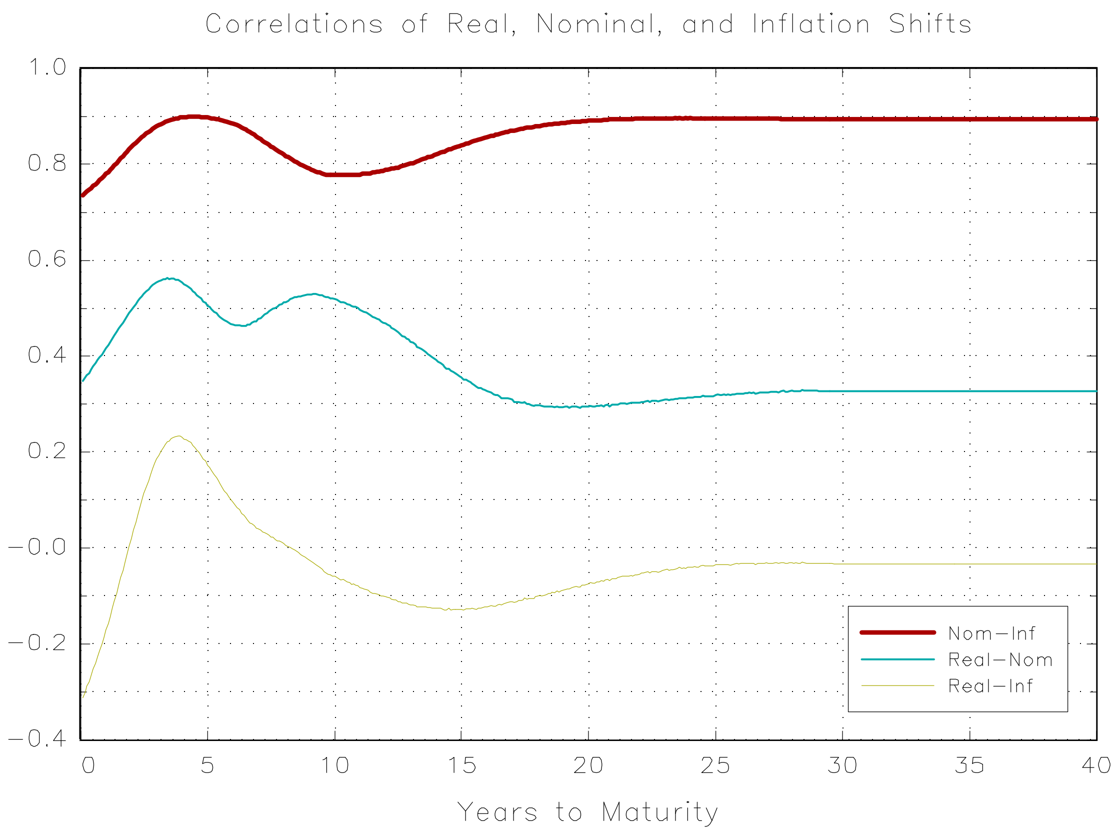

Figure 12.

Maturity-specific correlation coefficients between real forward rate shifts, nominal forward rate shifts, and inflation premium shifts.

Does the Long Forward Rate Ever Fall?

Dybvig, Ingersol and Ross (DIR, 1996) point out that if the (real or nominal) forward rate (and therefore the zero-coupon rate) has a limiting value as maturity approaches infinity, that limiting rate may never fall without generating arbitrage opportunities, in the absence of transactions costs. It must therefore either be nonstochastic or else nondecreasing over time. Intuitively, their argument is similar to the point made by Ingersol, Skelton and Weil (1978), that the yield curve cannot undergo shifts that are exclusively parallel without generating arbitrage opportunities. The same theorem is valid for exclusively parallel shifts of any portion of the forward curve, and, as DIR point out, even for the limiting value (if one exists) of the forward curve.[22] It follows that if the long-term rate has attained an ergodic distribution, that distribution can only be a degenerate, nonstochastic one.

The "natural" restriction at the long end of the QN spline requires that an asymptotic forward rate set in at the longest observed maturity. As already pointed out, it is not claimed here that longer-term bonds than those observed would in fact lie precisely on the thus extrapolated term structure, so that our estimated long-term rate may not be the true asymptotic rate. Nevertheless, if the forward curve is indeed heading for an asymptote, we would expect that the estimated long-term rate would give at least an "early warning" indication of its whereabouts.

It may be seen from Figure 10 above that the long-term real forward rate (beyond 30 years) is only slightly less volatile than rates under 13 years, and is in fact slightly more volatile than real forward rates 13-25 years out. The long-term nominal rate is in fact *more* volatile than any shorter maturity. Furthermore, Figures 5 and 6 demonstrate that the estimated long-term real and nominal forward rates (which have already set in by 1/1/2030 with this data) have no difficulty either falling *or* rising. It would therefore appear that if there is an asymptotic real or nominal forward rate obeying the DIR theorem, it has not yet even begun to set in at a horizon of 30 years.[23]

The real and nominal term structures shown here are based on bid-asked mean price quotations that do not take transactions costs into account. Separate bid and asked term structure

[22] McCulloch (2000b) points out that there is a crucial error in the basic theorem of DIR (1996), but that a simple modification of their proof restores their conclusion.

could easily be estimated, and would not differ by much at the maturities observed in the present paper. However, McCulloch (2000b) shows that in the presence of even very small fixed transactions costs, the bid and asked term structures eventually diverge without limit, making long-term forward rates indeterminate, so that perhaps the DIR proposition is not one of great practical significance.

Conclusions

Real term structures are fit to data from April 1998 to August 2000 by means of a Quadratic-Natural (QN) cubic spline functional form, developed here. This functional form is also fit, for comparison, to selected conventional nominal issues. Under risk-neutrality with respect to consumption uncertainty, as proposed by Stanley Fischer (1975), the reciprocal forward CPI implied by the real and nominal term structures should equal the market's expectation of the future purchasing power of money.

It is demonstrated that under the (Stanley) Fischer hypothesis of consumption-risk neutrality, the marginal inflation premium does not, in general, equal expected future inflation itself per the traditional (Irving) Fisher Equation, (1896, 1930), but rather is a weighted average of expected conditional future inflation rates, giving greatest weight to those cases in which future purchasing power is highest, and therefore in which future inflation is relatively low, in a modern fiat money economy.

It is found that nominal interest rate volatility is 2-2.5 times greater (in terms of standard deviation) than real interest rate volatility. The correlation between nominal interest rate shocks and inflation premium shocks lies in the range .8 to .9, so that about 60-80% of the variance in

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[23] McCulloch (2000b) also points out that in the presence of even small transactions costs, a true asymptotic

nominal interest rates is accounted for by the variance in the inflation premium. Real interest rate shocks have been essentially orthogonal to inflation premium shocks, but have a correlation of .3 to .5 with nominal interest rate shocks.

To date, there is no evidence for deviations from the log expectations hypothesis for real interest rates, or from either the Fisher hypothesis for the inflation premium The Fischer hypothesis is just barely rejected at the 95% level at a few short maturities that depend on an extrapolated yield curve. It remains to be seen if this rejection holds up with a longer time series.

The estimated long-term forward rate for 30 years and beyond is found to fall or rise with equal ease, and to have a volatility that is as large or even greater than shorter term forward rates. If there is an asymptotic interest rate obeying the Dybvig, Ingersoll and Ross (1996) theorem that the long-term rate cannot fall, it therefore does not yet appear to be showing signs of setting in at even a 30-year horizon.

The estimated term structures, forward CPI curves, and inflation premia are archived on the World Wide Web, and will be updated monthly.

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forward rate would in fact be indeterminate.

Appendix 1 — The Indexation Lag

US Treasury inflation-indexed securities are in fact indexed to the CPI-U with a short lag: The effective CPI for the first day of a given month is the actual CPI-U for the third month prior. Effective CPI values for other days of the month are obtained by linear interpolation. For example, the effective CPI for June 30, 1998, 162.49 (1982-4 = 100), is a linear interpolation of the actual CPI-U values for March (162.2) and April (162.5) of 1998. Payments of principal and interest are indexed by the ratio of the effective CPI for the due date to that for the date of issue. If a bond is purchased after issue, the quoted purchase price is by convention indexed by the ratio of the effective CPI for the purchase date to that for the date of issue. The U.S. Treasury's Office of the Public Debt maintains a website with these factors[24] Since the CPI-U reflects price information collected near the middle of the month in question, the effective indexation lag is approximately 2.5 months.

This lag effectively means that the last 2.5 months an indexed security is held is indexed by inflation during the 2.5 months prior to purchase. If inflation turns out to be constant, the quoted real yield will be the true yield regardless of the level of inflation. However, since inflation is uncertain, a small part of the return is in fact subject to inflation risk. Nevertheless, so long as the lag is much shorter than the life of the bond, the risk is very small compared to that on conventional nominal bonds.

The real term structures estimated here could, if desired, be fine-tuned by subtracting the shifted inflation premium back off the estimated nominal comparison term structure to obtain a lag-adjusted real forward curve, from which a lag-adjusted real zero coupon yield curve could be reconstructed using (3). This would in effect make the adjustment for changing inflationary

[24] At http://www.publicdebt.treas.gov/of/ofinflin.htm .

expectations called for by von Furstenberg and Gapen (1998), but using the forward inflation premium implicit in the two term structures in place of expected future inflation, in line with the recommendations of Deacon and Derry (1994b). Nevertheless, at current U.S. inflation rates and given the short indexing lag, the effects of such an adjustment would be quite small, and is not made in the present paper.[25]

The effects of lagged indexation are more important in the U.K., where an 8-month lag is used. Furthermore, the prices there are quoted in such a way that the last 8 months of each payment are not indexed at all, rather than by the inflation for the 8 months preceding the quotation, as would be the case under the American system with an 8-month lag. As a consequence, the real yield to maturity depends on the inflation that is assumed for this period, and yields are customarily quoted relative to more than one inflation assumption, eg 3% and 5%. U.S. real yield quotes require no such inflation assumption, since the preceding 2.5 months' inflation is automatically incorporated, for better or worse. A series of forward inflation rates can be inferred from the real and nominal U.K. term structures, and then used as the assumed inflation for the last 8 months of each payment in an iterative procedure, as in Deacon and Derry (1994b), but this is somewhat cumbersome.

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[25] Evans (1998) points out that even such a lag adjustment would not precisely give the real returns on hypothetical synchronously indexed bonds, and attempts to explicitly model of the premium for the residual inflation risk.

Appendix 2 — Details of Estimation

A bond with maturity m_i years and coupon rate c_i makes

$$h_i = \text{trunc}(2\ m_i) \qquad \text{(A-1)}$$

payments of $c_i/2$ before maturity, plus a payment of $1+c_i/2$ at maturity.[26] If its quoted "and-interest" price is p_i, it will actually trade at a "flat" price of

$$p_i^f = p_i + (.5 - (m_i - 2\ h_i))\ c_i \qquad \text{(A-2)}$$

that includes a pro-rated share of the next coupon payment. The pricing equation is then

$$p_i^f = \sum_{j=0}^{h_i-1} \frac{c_i}{2} d(j/2 + m_i - 2h_i) + (1 + c_i/2) d(m_i) \qquad \text{(A-3)}$$

The "and-interest" prices used here are the means of the bid and asked prices quoted in the *Wall Street Journal*.

Prior to 1969, there were strong tax effects in the U.S. bond market, as modeled by McCulloch (1975), but changes then in the tax treatment of capital gains on bonds purchased below par greatly reduced these effects, and the recent proliferation of tax-sheltered savings plans have further weakened the effect of taxation. Taxes are therefore not incorporated into the present model.

Linear combinations of the n+1 functions

$$\begin{aligned} q_1(m) &= m \\ q_2(m) &= m^2 \\ q_j(m) &= \max(0, m - m_{j-2})^3,\ j = 3, \ldots n+1 \end{aligned} \qquad \text{(A-4)}$$

[26] If $2m_i$ is an integer, this formula assumes there is a coupon at maturity 0, but this nets out of the flat price and has no effect on the pricing relationship.

generate the set of all cubic splines with knotpoints m_i that pass through the origin and are quadratic on their first interval. It follows that linear combinations of the n functions

$$y_j(m) = q_j(m) - \frac{q''_j(m_n)}{q''_{n+1}(m_n)} q_{n+1}(m), j = 1, \ldots n \tag{A-5}$$

are splines that pass through the origin, are quadratic on the first interval, and obey the "natural" restriction (10) at m_n. A QN Spline log discount function may therefore be constructed as

$$j(m) = \sum_{j=1}^{n} a_j y_j(m) \tag{A-6}$$

Any of a number of algorithms may be used to solve the n non-linear equations (A-3) for the n unknowns a_1, ... a_n. The method employed here begins by solving the n linear equations

$$m_i y_i = \sum_{j=1}^{n} a_j^0 y_j(m_i), i = 1, \ldots n \tag{A-7}$$

where y_i is the continuously compounded yield to maturity of the i-th security. Given a_j^q, the corresponding discount function

$$d^q(m) = \exp\left(-\sum_{j=1}^{n} a_j^q y_j(m)\right) \tag{A-8}$$

is used to evaluate the coupons to arrive at a net (of coupons) price

$$p_i^{net(q)} = p_i^f - \sum_{j=0}^{h_i - 1} \frac{c_i}{2} d^q(j/2 + m_i - 2h_i) \tag{A-9}$$

Then

$$\log\left((1 + c_i/2)/p_i^{net(q)}\right) = \sum_{j=1}^{n} a_j^{q+1} y_j(m_i), i = 1, \ldots n \tag{A-10}$$

may be solved linearly for a_j^{q+1}, etc. This was repeated until the zero-coupon yield curve converged to within 0.001 percentage point at each of 481 maturities from 0 out to 40 years.

This method required 16-27 iterations for the present examples and takes only a second or two per data set on a Pentium-100 in GAUSS.[27]

The exact QN spline lends itself to observations whose terminal maturities are widely spaced, proportionately speaking. Observations whose terminal maturities are close to one another may easily generate wild swings in the forward curve arising from transactions costs alone. However, closely spaced data may be accommodated by fitting a spline log discount function with far fewer parameters than observations, by nonlinear least squares. In such a case, the knots may be placed with equal numbers of terminal maturities between knots, as in McCulloch (1971, 1975). The number of parameters may either be set to the nearest integer to the square root of the number of observations, as in McCulloch (1971, 1975), or, better, may be determined by a criterion of no significant serial correlation in the residuals.[28] The natural restriction at the long end of the log discount function should be retained. The quadratic restriction at the short end is no longer necessary with a regression spline, and may either be dropped by adding a cubic term to (A-4) (to obtain what might be called a "semi-natural spline"), or retained.

[27] With very long maturities and/or very high interest rates, there is an off chance that the estimated net price will be negative and the algorithm described will fail. It performs well for the data employed here, however.

[28] A third alternative would be to add knots by iteratively bisecting (in terms of data points) that interval in which serial correlation in the pricing errors is most evident, until there is no statistically significant overall serial correlation.

REFERENCES

Adams, Kenneth J. and Donald R. Van Deventer, "Fitting Yield Curves and Forward Rate Curves with Maximum Smoothness." *Journal of Fixed Income* June 1994, pp. 52-62.

Bidarkota, Prasad V., and J. Huston McCulloch, "Optimal Univariate Inflation Forecasting with Symmetric Stable Shocks," *Journal of Applied Econometrics,* forthcoming 1998.

Bliss, Robert R. "Testing Term Structure Estimation Methods," *Advances in Futures and Options Research,* Vol. 9, JAI Press, 1997, pp. 197-231.

Breedon, Francis J., and Jagjit S. Chadha, "The Information Content of the Inflation Term Structure," Bank of England Working Paper # 75 (1997).

Cox, John, Jonathan E. Ingersoll, Jr., and Stephen A. Ross, "A Reexamination of Traditional Hypotheses about the Term Structure of Interest Rates," Journal of Finance 41 (1981): 769-99.

Deacon, Mark, and Andrew Derry, "Estimating the Term Structure of Interest Rates," Bank of England Working Paper # 24, 1994a.

_____________, "Deriving Estimates of Inflation Expectations from the Prices of UK Government Bonds," Bank of England Working Paper # 23, 1994b.

Duffie, Darrell, "Special Repo Rates," *J. Finance* **51** (1996): 493-526.

Dybvig, Philip H., Jonathan E. Ingersoll, Jr., and Stephen A. Ross, "Long Forward and Zero-Coupon Rates Can Never Fall," *Journal of Business* **69** (1996): 1-24.

Evans, Martin D.D., "Real Rates, Expected Inflation and Inflation Risk Premia," *Journal of Finance* **53** (Feb. 1998): 187-218.

Fischer, Stanley, "The Demand for Indexed Bonds," *Journal of Political Economy* **83** (1975): 509-34.

Fisher, Irving, "Appreciation and Interest," *Publications of the American Economic Association* **9**, (1896): 509-34.

___________. "The Theory of Interest," 1930. Reprinted by Augustus M. Kelley, 1965.

Fisher, Mark, Douglas Nychka, and David Zervos, "Fitting the Term Structure of Interest Rates with Smoothing Splines," Finance and Economics Discussion Series, Division of Monetary Affairs, Federal Reserve Board, Jan. 1995.

von Furstenberg, George, and Michael Gapen, "Conditional Indexation Bias in Yields Reported on Inflation-Indexed Securities, with Special Reference to Udibonos and TIPS," *conomía Mexicana* **NE 7**, 1998, 149-88.

Ingersoll, Jonathan E., Jr., Jeffrey Skelton, and Roman L. Weil, “Duration Forty Years Later,” *Journal of Financial and Quantitative Analysis* **13** (1978): 627-48.

Klein, Benjamin, "Our New Monetary Standard: The Measurement and Effects of Price Uncertainty 1880-1973," *Economic Inquiry* **13** (1975): 461-83.

Kochin, Levis A. "Interest Rates and Uncertain Future Price Levels," University of Washington Working Paper, 1980.

McCulloch, J. Huston, "Measuring the Term Structure of Interest Rates," *Journal of Business* **44**, 1971, 19-31.

___________. "An Estimate of the Liquidity Premium," *Journal of Political Economy* **83**, January/February 1975a, pp. 95-119.

___________. "The Tax Adjusted Yield Curve," *Journal of Finance* **30**, June 1975b, 811-30.

___________. "The Cumulative Unanticipated Change in Interest Rates: Evidence on the Misintermediation Hypothesis," NBER Working Paper # 222, Dec. 1977.

___________. "The Monotonicity of the Term Premium: A Closer Look." *Journal of Financial Economic*s **18**, March 1987, pp. 185-91.

___________. "A Reexamination of Traditional Hypotheses about the Term Structure: A Comment," *Journal of Finance* **47** (1993): 779-89.

___________. "Financial Applications of Stable Distributions," in G.S. Maddala and C.R. Rao, eds., *Statistical Methods in Finance* (vol. **14** of *Handbook of Statistics*), Elsevier, 1996, pp. 393-425.

___________. “Long Forward and Zero-Coupon Rates Indeed Can Never Fall, but Are Indeterminate: A Comment on Dybvig, Ingersoll and Ross,” MS, August 1999.

___________. “The "Bellwether" 30-Year Treasury Bond is an Exceptionally Bad Investment,” Feb. 2000, at http://www.econ.ohio-state.edu/jhm/ts/otr.htm.

___________ and Heon-Chul Kwon, "US Term Structure Data, 1947-1991," Ohio State University Economics Dept. Working Paper #93-6. Available with data at http://economics.sbs.ohio-state.edu/jhm/jhm.html .

Robinson, Joan, "The Rate of Interest," *Econometrica* **19** (1951): 92-111.

Shiller, Robert J., "The Volatility of Long-Term Interest Rates and Expectations Models of the Term Structure," *Journal of Political Economy* **87** (1979): 1190-1219.

Stec, Jeffrey A., “Money Demand and the Moderate Quantity Theory of Money: An Empirical Investigation,” Unpublished Ph.D. dissertation, Ohio State University, September 2000.

Vasicek, Oldrich A., and H. Gifford Fong, "Term Structure Modeling Using Exponential Splines," *Journal of Finance* **37** (1977): 339-56.

Waggoner, Daniel. "Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices," MS, Federal Reserve Bank of Atlanta, Oct. 10, 1997.