Long Forward and Zero-Coupon Rates Indeed Can Never Fall,

but Are Indeterminate:

a Comment on Dybvig, Ingersoll and Ross

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Abstract

In a recent paper, Dybvig, Ingersoll and Ross claim to have proven that in the absence of transactions costs, the long-term forward and zero-coupon interest rates may rise, but never may fall.

This comment shows that there is a crucial error in the proof of their basic theorem, but that a modification of their proof restores their conclusion.

The comment goes on to show that in the presence of even very small transaction costs, the long–term zero-coupon and forward interest rates are undefined and should not be taken literally.

In a recent paper in this *Journal*, Dybvig, Ingersoll and Ross (DIR, 1996) claim to have proven that in the absence of transactions costs, the long-term forward and zero-coupon interest rates may rise, but never may fall.

This comment shows that there is a crucial error in the proof of their basic theorem, but that a modification of their proof restores their conclusion. A further problem is that the DIR definition of an arbitrage opportunity is excessively broad, but this defect does not affect their theorem, and can easily be patched up.

The comment goes on to show that in the presence of even very small transaction costs, the long–term zero-coupon and forward interest rates are in fact undefined and should not be taken literally.

The DIR Theorem

In DIR's notation, $n(t,T)$ is the price at time t of a real or nominal 1 unit payoff at a later date T. In terms of the annual compounding used by DIR, the yield to maturity on such a zero-coupon loan is given by:

$$z(t,T) = n(t,T)^{-1/(T-t)} - 1.$$

If the limit exists, the long-term zero-coupon yield is

$$z_L(t) = \lim_{T\uparrow\infty} z(t,T) .$$

The DIR Theorem 2 states that "The Long Zero-Coupon Rate Can Never Fall." Two proofs are provided, a simple one in the text for the case of a finite number of future states of the world, and a more complicated one in an Appendix for the case of a continuum of future states.

In the finite state case treated in their text, DIR let ω represent a typical state that may occur at future time s > t. The future zero-coupon rate is $z(s,T;\omega)$ if state ω occurs, and $z_L(s;\omega)$ is the corresponding long-term zero-coupon rate. The latter achieves its smallest value, $z_L(s;\omega^*)$, in state ω*, which has probability that is positive, yet less than unity. DIR assume, contrary to the Theorem, that

$$z_L(t) > z_L(s;w^*), \quad (1)$$

and then attempt to show that an arbitrage opportunity exists.

To do this, they consider a trade consisting of buying, at current date t, $[1+z_L(s;\omega^*)]^{T-s}$ units of payment maturing at future date T for

$$\frac{[1+z_L(s;w^*)]^{T-s}}{[1+z(t,T)]^{T-t}}, \quad (2)$$

and then selling this position at intermediate date s for

$$\frac{[1+z_L(s;w^*)]^{T-s}}{[1+z(s,T;w)]^{T-s}}. \quad (3)$$

They argue a) that (2) [their (10)] tends to 0 as $T \uparrow \infty$ because of (1), b) that (3) [their (11)] tends to 1 in states for which $z_L(s;\omega) = z_L(s;\omega^*)$, and c) that (3) tends to 0 in states for which $z_L(s;\omega) > z_L(s;\omega^*)$. They conclude that this would lead to an arbitrage opportunity, since an essentially costless investment would have a positive payoff with positive probability and no chance of a negative payoff, and that therefore (1) must be false, so that

$$z_L(t) = z_L(s;w^*). \quad (4)$$

The DIR Error

However, b) is not warranted under the assumptions stated, since the limit in question depends critically on the rate of convergence of the zero-coupon rate to its limit. The basic fallacy in their reasoning is the false assumption that $\lim_{x\uparrow\infty} f(x) = 0$ implies $\lim_{x\uparrow\infty} xf(x) = 0$.

For example, suppose that for large T,

$$\log(1+z(s,T;w)) = \log(1+z_L(s;w)) + a(T-s)^b$$

for some coefficients a and b, with b < 0. Then the time s zero-coupon rate converges to its long rate in every state, yet (3) becomes

$$\exp\left[(T-s)\log(1+z_L(s,w^*)) - (T-s)\log(1+z_L(s;w)) - a(T-s)^{b+1}\right].$$

This tends to unity in states for which $z_L(s;\omega) = z_L(s;\omega^*)$ only if a = 0 or b < -1. If a > 0 and b > -1, it tends instead to 0 in these states, while if a < 0 and b > -1, it tends to +∞.

Since there is no guarantee that the trade will have a positive payoff, no arbitrage opportunity has been demonstrated, and the DIR proof is defective.

An Alternative Proof

Despite the error in DIR's proof of their Theorem, it may easily be patched up by considering instead the purchase at time t of $[1+z(s,T;\omega^*)]^{T-s}$ units to be repaid at T, but sold before maturity at date s. The time t cost of this investment is

$$\frac{[1+z(s,T;w^*)]^{T-s}}{[1+z(t,T)]^{T-t}},$$

which still tends to 0 as T ↑ ∞, yet its value at date s,

$$\frac{[1+z_L(s,T;w^*)]^{T-s}}{[1+z(s,T;w)]^{T-s}},$$

is identically unity when ω = ω*. Equation (1) therefore does indeed create an arbitrage oportunity, and hence (4) must be true.

Equation (4) implies that the long rate takes on its lowest possible future value, and therefore may not fall further. Because the long-run forward rate, if it exists, must equal the long-run zero-coupon rate, it then follows that it may not fall either. Furthermore, if these rates have an ergodic distribution, it must be the trivial nonstochastic distribution.

Arbitrage Opportunities

Although it does not affect the substance of their paper, it should be noted that the DIR definition of an arbitrage opportunity is overly general. They define an arbitrage opportunity to include any sequence of trades in which the price tends to zero but the payoff tends uniformly to a nonnegative random variable that is positive with positive probability. This definition would include a sequence of trades involving buying at time t a unit zero-coupon bond maturing at time T_i for price $n(t, T_i)$, and holding it to maturity. For $T_i = 1, 2, \ldots \infty$, the price of this investment goes to 0 even though its payoff at T_i is unity in all states of the world, yet it obviously does not constitute an arbitrage opportunity.

The DIR theorem in fact calls for cashing in the investment at a fixed future date s short of infinity, but even then the definition is too broad. Consider, for example a world in which the marginal utility of output is constant for the representative agent at all horizons, unless the earth is struck by an asteroid that extinguishes all life, in which case it falls to zero. If the occurrence of this event between time t and T has the exponential probability $1 - e^{-r(T-t)}$, the real term structure will be flat at rate ρ. Suppose that a technology exists that enables one loaf of bread to be rocketed into space and automatically returned to earth in perfect condition at any future date s, whether or not this event has occurred in the meantime. The value at time t of a conditional claim on such an asset, conditional on this event having occurred, is 0 regardless of s, yet it will indeed pay one unit of output in this positive probability state. [1]

To preclude such cases, the DIR definition of a positive arbitrage opportunity should be revised to encompass sequences of trades in which the price tends to zero, yet the payoff tends to a nonnegative value, bounded away from 0, in states in which output has a positive state-contingent present value, likewise bounded away from 0. A similar modification can be made for negative arbitrage opportunities.

The Long Rate with Transactions Costs

Although it is true that in a world of zero transactions costs and complete markets, the long term real or nominal interest rate may not fall, and therefore must be nonstochastic if ergodic, in the presence of even small transactions costs, the long-term rate is indeterminate, and therefore its behavior is in fact economically moot.

Suppose, for example, that the time t bid price of a claim on one unit of output at future date T is

$$n^{bid}(t,T) = n(t,T) - e,$$

while the time t asked price of the same claim is

$$n^{ask}(t,T) = n(t,T) + e,$$

for some nonnegative function $n(t,T)$ that decreases in T toward 0 and some small transaction cost ε > 0.

[1] Somewhat less cataclysmically, one could consider a nominal claim in a world in which the price level is constant, barring a Poisson-driven complete collapse of the value of paper money, while the real interest rate is a constant.

Wall St. Journal quotations on long-term nominal U.S. Treasury STRIPS indicate a bid-asked spread (2ε) of 4/32 per $100 of face value, implying ε = .000625. Inter-dealer spreads are smaller, but are still non-zero. In practice, this spread tends to increase in dollar terms, if anything, with maturity.

Such a spread implies separate bid and asked zero-coupon yield curves, with the properties that $\lim_{T \uparrow \infty} z^{ask}(t,T) = 0$, while $z^{bid}(t,T)$ rises to infinity as $n(t,T)$ falls to ε. If $n(t,T) = e^{-.06(T-t)}$ and ε = .000625, the latter would occur at T = t + 123 years. It can be shown that if the zero coupon yield curve is inferred from quotations on coupon bonds selling near par, its indeterminacy increases even more quickly with maturity.

In the presence of even very small transactions costs, long-term zero-coupon rates, and therefore long-term forward rates, are therefore indeterminate and therefore cannot have a true limit as T ↑ ∞. The DIR theorem, as corrected here, that such a limiting rate cannot fall, is certainly an interesting proposition, yet one with questionable practical significance.[2]

It should be noted that although the long-term zero-coupon and forward interest rates are indeterminate in the presence of fixed transactions costs, the same is not ordinarily true of the long-term par bond yield.[3]

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[2] McCulloch and Kochin (1999) fit a “QN Spline” to the log discount function that ties into a forward curve that is flat from the longest observed maturity to infinity. This should not be interpreted as a literal estimate of the forward rate at infinite maturity, but merely as a plausible estimate of the behavior of forward rates in the immediate vicinity of the longest observed maturity.

[3] As DIR demonstrate in their equation (8), the par bond yield of any maturity is equal to a weighted average of the intervening forward rates, where the weights are the corresponding discount factors. Therefore the par bond yield curve may have a long term limit even if forward rates and zero coupon rates are increasingly ill defined as maturity increases.

REFERENCES

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