Econ 741 (Spring 2003): Suggested Answer for Midterm

< **Question 1** >

a) Price elasticity of rose demand is -0.301 and cross price elasticity of rose demand is -0.102. In other words, on the average, the demand for rose falls 0.301 % as price of rose increases 1 % and it also falls 0.102 % as price of carnation rises 1 % when other things are constant.

b) The critical value is | 1.721 |. As t-value is -0.342/0.162 = -2.111 < -1.721, we reject the null hypothesis.

c) The standard error is 0.139. From a t-distribution table for d.f=20, $t_{0.025}$ = 2.086. So [-0.301 - (0.139)(2.086) $\leq \beta_2 \leq$ -0.301 + (0.139)(2.086)]. Hence the 95% confidence interval is [-0.5909,-0.011]

d) Because we expect the demand for rose to increase when the price of carnation rises based on substitute argument, $H_1 : \beta_3 > 0$ makes sense. Since the sign is incorrect, we fail to reject the null hypothesis.

e) Using a one-tailed test for $H_1 : \beta_3 > 0$, the critical value is 1.325 for the 10 % significant level and the t-value is 0.102/0.412 = 0.247. Hence we can't reject the null hypothesis.

f) The F test statistic is $F = \frac{R^2}{(1-R^2)}\frac{n-k}{k-1} = \frac{0.678}{1-0.678}\frac{20}{2} = 21.05$. The critical value is 3.49. Hence we reject the null hypothesis.

g) For regression (1), $\overline{R}^2 = 1 - (1 - R^2)\frac{n-1}{n-k} = 1 - (1 - 0.627)\frac{22}{21} = 0.6092$. For regression (2), $\overline{R}^2 = 1 - (1 - R^2)\frac{n-1}{n-k} = 1 - (1 - 0.678)\frac{22}{20} = 0.6458$. In terms of $\overline{R}^2$, regression (2) is preferred.

< **Question 2** >

a) Under no multi-collinearity assumption, $b = Ay$, where $A = (X'X)^{-1}X'$. With linear assumption, $b = A(X\beta + \epsilon) = \beta + A\epsilon$. E$(b \mid X) = \beta + AE(\epsilon \mid X)$. Due to strict exogeneity assumption, $E(b \mid X) = \beta$. Spherical error variance assumption is unnecessary for this result.

b) $Var(b \mid X) = Var(b - \beta \mid X) = Var(A\epsilon \mid X) = AVar(\epsilon \mid X)A' = AE(\epsilon\epsilon' \mid X)A' = A(\sigma^2 I_n)A' = \sigma^2 AA' = \sigma^2(X'X)^{-1}$.

c) $\hat{\beta} = Cy = (D + A)y = Dy + Ay = D(X\beta + \epsilon) + b = DX\beta + D\epsilon + b$. Taking conditional expectation of both sides, we obtain $E(\hat{\beta} \mid X) = DX\beta + E(D\epsilon \mid X) + E(b \mid X)$. Since both b and $\hat{\beta}$ are unbiased and since $E(D\epsilon \mid X) = DE(\epsilon \mid X) = 0$, it follows that $DXB = 0$

d) $DX\beta = 0$ holds for all β. To show this, let $DX = Z$, where Z is (k x k) matrix. Let $\beta = (1, 0, \cdots, 0)'$, then $Z\beta = Z_{i1} = 0$, where $i = 1, \cdots, k$. Further, let $\beta = (0, 1, \cdots, 0)'$, then $Z\beta = Z_{i2} = 0$. Similarly, let $\beta = (0, 0, \cdots, 1)'$, then $Z\beta = Z_{kk} = 0$. Therefore, $DX = Z = (Z_{i1}, \cdots, Z_{kk}) = 0$.

e) From **c**) and **d**), $\hat{\beta} = D\epsilon + b$. Subtracting β from both sides produces $\hat{\beta} - \beta = D\epsilon + (b - \beta) = (D + A)\epsilon$. Therefore, $Var(\hat{\beta} \mid X) = Var(\hat{\beta} - \beta \mid X) = Var((D + A)\epsilon \mid X) = (D + A)Var(\epsilon \mid X)(D' + A') = \sigma^2(D + A)(D' + A') = \sigma^2(DD' + AD' + DA' + AA')$

f) Since $AD' = D'A = 0$ and $AA' = (X'X)^{-1}$ in **e**), $Var(\hat{\beta} \mid X) = \sigma^2[DD' + (X'X)^{-1}] = \sigma^2(X'X)^{-1} + \Delta$, where $\Delta(= \sigma^2(DD'))$ is a positive semidefinite matrix.

< **Question 3** >

a) $trace$(M)= $trace(I_n - (X(X'X)^{-1}X'))$ = $trace(I_n)$ - $trace((X'X)^{-1}X'X)$ = $trace(I_n)$ - $trace(I_k)$ = n-k

b) First, define two random variables z_k and q and their distributions. Let $z_k = \frac{b_k - \beta_k}{\sqrt{\sigma^2[(X'X)^{-1}]_{kk}}}$. Because $b \mid X = N(\beta, \sigma^2(X'X)^{-1})$, $z_k \mid X = N(0,1)$. Let $q = \frac{\epsilon' M \epsilon}{\sigma^2}$. Because $q = (\frac{\epsilon}{\sigma})' M (\frac{\epsilon}{\sigma})$ and $rank$(M) = $trace$(M) = n-k from Fact 1.4, $q \mid X \sim \chi^2(n-k)$ from Fact 1.3.

Second, show z_k and q are independent conditional on X. Note that z_k is a function of b and q is a function of e. So we have to show b and e are independently distributed conditional on X. Both b and e are linear functions of ϵ, so they are jointly normally distributed conditional on X. Because $b = \beta + (X'X)^{-1}X\epsilon$, $e = M\epsilon$ and $MX = 0$, each element of b is uncorrelated with each element of e. Hence they are independently distributed conditional on X.

Finally, from Fact 1.2, $t_k = \frac{z_k}{\sqrt{q/(n-k)}}$ is distributed as a t random variable with $(n-k)$ degrees of freedom conditional on X.

c) Because the conditional distribution does not depend on X, the unconditional distribution is the same.

< **Question 4** >

Let $\hat{\sigma^2} = \frac{e'e}{n}$. Then, $E(\hat{\sigma^2} \mid X) = E(\frac{e'e}{n} \mid X) = \frac{1}{n}E(e'e \mid X)$. Since

$E(e'e \mid X) = \sigma^2(n-k)$, $E(\hat{\sigma^2} \mid X) = \frac{1}{n}\sigma^2(n-k) = \sigma^2 - \frac{\sigma^2 k}{n} \neq \sigma^2$.

< **Question 5** >

a) The GLS is obtained by applying the OLS to the transformed model $\frac{y_t}{x_t} = \frac{\beta_1}{x_t} + \beta_2 + \frac{\epsilon_t}{x_t}$. Then $Var(\frac{e_t}{x_t}) = \sigma^2$ and the Gauss-Markov theorem applies to the transformed model. Thus, the GLS estimator can be computed by regressing $\frac{y_t}{x_t}$ onto $\frac{1}{x_t}$ and constant. The coefficient for the $\frac{1}{x_t}$ term is the GLS estimator for β_1 and the coefficient for constant is the GLS estimator for β_2.

b) $R^2_{OLS} \geq R^2_{GLS}$.

By construction, $\hat{\beta_{OLS}} = arg.min\ SSR = arg.min\ TSS(1-R^2) = arg.min(1-R^2) = arg.max\ R^2$

< **Question 6** >

a) P'=[X(X'X)$^{-1}$X']'=X(X'X)$^{-1}$X'=P. Thus P is symmetric. Since I$_n$ and P are symmetric, M(=I$_n$ - P) is also symmetric. P^2=[X(X'X)$^{-1}$X'][X(X'X)$^{-1}$X'] =X(X'X)$^{-1}$X'=P, P is idempotent. M^2=[I$_n$ - P][I$_n$ - P] = I$_n$ - 2P + P^2 = I$_n$ - P = M, M is also idempotent.

b) PX = [X(X'X)$^{-1}$X']X = X. Thus, P is the projection matrix

c) MX = (I$_n$ - P)X = X - PX = X - X = 0. Thus, M is the annihilator.