The Texas Problem

Maoyong's Foreign Devil Boss is so happy that Maoyong gets promoted to the boss of the whole American Multinational Corporation. Now his new problem is: What happens if our company has to use up all the oil, but we can pick our own time to end the problem? Maoyong now relaxes the constraint that T is given. The equations governing the dynamics of $u(t)$ and $y(t)$ along the optimal trajectory are unchanged. In particular:

$$y^*(t) = \hat{c} + \frac{\lambda^*}{2\rho}e^{\rho t} - \frac{t}{2}$$

As in the case of the American problem, we continue to require that:

$$\lambda(T) \geq 0,\ y(T) \geq 0,\ \lambda(T)y(T) = 0$$

We must now include the additional transversality condition:

$$H(T, y(T), u(T)) - \lambda(T)\left(\frac{dy}{dt}\right)_{\Phi=0} = 0$$

The term, $\left(\frac{dy}{dt}\right)_{\Phi=0}$, allows for the possibility of a trade-off between extending the terminal time and altering the terminal value. In this case, there is no such tradeoff, and so we may presume that $\left(\frac{dy}{dt}\right)_{\Phi=0} = 0$. Our original specification of H may be evaluated at $t = T$, to obtain:

$$H(T) = u(T)[(1 - u(T))e^{-\rho T} - \lambda(T)]$$

The additional transversality condition for this case requires that $u(T) = 0$. However, $u(t) = -\dot{y}(t)$. Therefore, the transversality condition on T requires that:

$$u^*(T) = \frac{1}{2} - \frac{\lambda^*}{2}e^{\rho T} = 0$$

Therefore, the time constant value of the costate variable, λ, must be given by::

$$\lambda^* = e^{-\rho T}$$

There are two possibilities. We might have $T^* \to +\infty$, and $\lambda^* = 0$, or else $\lambda^* > 0$. For $\lambda^* > 0$, we obtain:

$$100 = \hat{c} + \frac{\lambda^*}{2\rho}$$

$$0 = \hat{c} + \frac{\lambda^*}{2\rho}e^{\rho T^*} - \frac{T^*}{2}$$

$$\lambda^* = e^{-\rho T^*}$$

The optimal value of T, $T = T^*$, solves:

$$(1 - e^{-\rho T^*}) = \rho(T^* - 200)$$

Alternatively, suppose that $\lambda^* = 0$. In this case, $y^*(t) = 100 - \frac{t}{2}$. Therefore $y^*(T^*) \to -\infty$, and so the constraint on $y(T)$ is violated. Therefore, we cannot have a solution with $\lambda^* = 0$, and so our earlier conjecture that $\lambda^* > 0$ must have been correct. Maoyong has therefore solved the most general form of the problem. He dies a very happy man, with lots of money and almost perfect wisdom.