Saddlepath Problem

Standard Representation:

$$\dot{x}_1 = -2x_1$$
$$\dot{x}_2 = x_1 + 3x_2$$

Matrix - Vector Representation:

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 \\ 1 & 3 \end{bmatrix}\mathbf{x}$$

Characteristic Equation:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \Rightarrow \begin{vmatrix} -2-\lambda & 0 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$-(2+\lambda)(3-\lambda) = 0 \Rightarrow \lambda_1 = -2, \lambda_2 = 3$$

Eigenvectors:

$$\begin{bmatrix} \mathbf{A} - \lambda_1\mathbf{I} \end{bmatrix}\mathbf{v}_1 = 0 \Rightarrow \mathbf{v}_1 = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{A} - \lambda_2\mathbf{I} \end{bmatrix}\mathbf{v}_2 = 0 \Rightarrow \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solution to Differential Equation:

$$\mathbf{x}(t) = c_1e^{-2t}\begin{bmatrix} 5 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Initial Conditions:

$$\mathbf{x}(0) = c_1\begin{bmatrix} 5 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$
$$= \begin{bmatrix} 0.2 & 0 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0.2x_1(0) \\ 0.2x_1(0) + x_2(0) \end{bmatrix}$$

Finally,
$x_1(t) = x_1(0)e^{-2t}$
$x_2(t) = -0.2x_1(0)e^{-2t} + 0.2(x_1(0) + 5x_2(0))e^{3t}$

Saddlepath requires that:
$x_1(0) + 5x_2(0) = 0.$

Suppose that $x_1(0) = 8,$ fixed.
$x_2(0)$ free.
Saddlepath requires that $x_2(0) = -0.2x_1(0) = -\frac{8}{5}$

Phase Diagram

Drawn in x_1, x_2 space.

$\dot{x}_1 = 0$ locus

$\dot{x}_1 = -2x_1 = 0 \Rightarrow x_1 = 0.$
Straight line vertical through the origin.
$\frac{\partial \dot{x}_1}{\partial x_1} = -2 < 0.$ Motion is toward the $\dot{x}_1 = 0$ locus.

$\dot{x}_2 = 0$ locus

$\dot{x}_2 = x_1 + 3x_2 = 0 \Rightarrow x_2 = -\frac{1}{3}x_1$
Downward sloping through the origin.
$\frac{\partial \dot{x}_2}{\partial x_2} = 3 > 0.$ Motion is away from the $\dot{x}_2 = 0$ locus.

The saddlepath will lie on the stable eigenvector: $\begin{bmatrix} 5 \\ -1 \end{bmatrix}$.

Initial conditions lying on the stable eigenvector imply stable paths.
Initial condition off the stable eigenvector imply unstable paths.