The Problem of Maoyong

Maoyong is fresh out of graduate school at a leading U.S. University with an Economics Ph.D. He now goes home and lands a prestigous government job. His new employer, the Prime Minister asks him to solve a problem. His new boss says, "act as if you own all of the oil in China, and that you are to sell all of this oil over a fifty year time span." Maoyong estimates that China's production is a large enough fraction of world output that he faces a downward-sloping demand curve. That is, he may act as a monopolist. He estimates the demand curve for oil to be given by $P(\text{Oil}) = \left[1 - (\text{Quantity Sold})\right]$, with quantities stated in Trillions of barrels. He discounts future oil revenue at the rate ρ. Presumably ρ is equal to the world real interest rate. Also assume that he starts with 100 Trillion barrels in the ground and that extraction costs are minimal. Denote the amount of oil sold as $-\frac{dy}{dt}$, where $y(t)$ denotes the amount of oil left in the ground. Oil revenue is therefore given by $-\frac{dy}{dt}(1-(-\frac{dy}{dt}))$. Therefore, Maoyong must solve the following problem:

$$\underset{y(t)}{Max} \int_0^{50} [(-\dot{y}(1+\dot{y})e^{-\rho t}]dt,$$

$$y(0) = 100, \;\; y(50) = 0.$$

If Maoyong fails, then the Party Directorate will send him to be an elementary school teacher in the Gobi desert.

Maoyong first needs to specify the problem in a manner that allows application of the Maximum Principle. He choses the amount of oil in the ground as the state variable, $y(t)$. He also chooses the control variable, $u(t)$, to be rate of sales of oil. Sales of oil deplete the stock of oil in the ground, and so:

$$\dot{y} = \frac{dy}{dt} = -u(t)$$

The problem has been reformulated as:

$$\underset{u(t)}{Max} \int_0^{50} [(u(t)(1-u(t))e^{-\rho t}]dt,$$

subject to :

$$\dot{y}(t) = -u(t), \, y(0) = 100 \, , \, y(50) = 0.$$

The Hamiltonian for this problem is given by:

$$H(t,y(t),u(t),\lambda(t)) = [(u(t)(1-u(t))e^{-\rho t}] - \lambda(t)u(t)$$

Recall the first-order conditions for the optimal control problem. At an interior solution, it must be true that:

$$\frac{\partial H}{\partial u} = [1-2u(t)]e^{-\rho t} - \lambda(t) = 0$$
$$\dot{\lambda}(t) = \frac{d\lambda}{dt} = -\frac{\partial H}{\partial y} = 0$$

The second of these conditions tells us that λ must be time-constant. Denote this constant as λ^*. Now use the first of these conditions to derive:

$$[1-2u^*(t)]e^{-\rho t} = \lambda^*, \text{ and so:}$$
$$u^*(t) = \frac{1}{2} - \frac{\lambda^*}{2}e^{\rho t}$$

Although we know that λ^* is time-constant, we do not yet know its precise value. Recall that $y(t)$ is governed by:

$$\dot{y}^*(t) = -u^*(t) = -\frac{1}{2} + \frac{\lambda^*}{2}e^{\rho t}$$

Integrating this expression with respect to time solves this simple first-order differential equation. The solution is of the form:

$$y^*(t) = \hat{c} + \frac{\lambda^*}{2\rho}e^{\rho t} - \frac{t}{2}$$

So Maoyong is halfway back to Peking now. How can he determine the constants? The equation above has two unspecified constants, $\hat{c}$ and λ^*. These constants may be solved via the initial and terminal conditions, $y(0) = 100$ and $y(50) = 0$. He knows that:

$$y_0 = 100 = \hat{c} + \frac{\lambda^*}{2\rho}$$

and

$$y_{50} = 0 = \hat{c} + \frac{\lambda^*}{2\rho}\exp(\rho 50) - 25$$

He tells the Prime Minister that these are two linear equations in the two constants. So $\hat{c} = -75/(1-\exp(\rho 50))$ and $\lambda^* = 150\rho/(1-\exp(\rho 50))$! Maoyong is a happy fellow, and he gets promoted to be the head of an American corporation in China.