Eigenvalues and Eigenvectors

For second-order and higher-order differential equations, it is often helpful to express the equation in a matrix-vector form. In order to work with such systems, we need to have a working knowledge of some facts from linear algebra.

Let **A** represent an $n \times n$ matrix, let **x** represent an $n \times 1$ vector, and let λ represent a scalar. An eigenvalue, λ_i, is a solution to:

$$|\mathbf{A} - \lambda \mathbf{I}| = 0,$$

where **I** denotes the identity matrix:

$$\mathbf{I} \equiv \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Suppose that **A** is given by:

$$\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The eigenvalues are therefore solutions to the polynomial obtained by evaluating the determinant:

$$\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} = 0$$

If A is an $n \times n$ matrix, there will be n values of λ that satisfy the characteristic equation, $|\mathbf{A} - \lambda \mathbf{I}| = 0$.

The solutions, λ_i, come in three varieties. Each, or some of the eigenvalues may be distinct. Each, or some of the eigenvalues may be repeated. Each, or some of the eigenvalues may occur in complex conjugate pairs. Repeated eigenvalues occur when the characteristic equation can be factored in a form that include terms of the form $(1 - \lambda_i)^m$. In this case the eigenvalue, λ_i, is repeated m times. Complex conjugate pairs occur when the characteristic equation can be factored in a form that includes one or more terms of the form, $((a - \lambda^2) + b^2)$. Some other interesting properties of the eigenvalues include that:

$$\lambda_1\lambda_2\cdots\lambda_n = |\mathbf{A}| \text{ and } \lambda_1 + \lambda_2 + \ldots + \lambda_n = trace(\mathbf{A})$$

As an example, suppose that n is equal to 2. In this case, the characteristic equation is given by:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$$

Define $\alpha \equiv -(a_{11}+a_{22})$ and $\beta \equiv (a_{11}a_{22} - a_{12}a_{21})$. The eigenvalues therefore given by:

$$\lambda_i = -\frac{\alpha}{2} \pm \sqrt{\left(\frac{\alpha}{2}\right)^2 - \beta}$$

For each of the n eigenvalues, λ_i, there corresponds an eigenvector, $\mathbf{v}_i$. Eigenvectors satisfy the equation:

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i, \text{ or: } (\mathbf{A} - \lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$$

Because $\mathbf{A} - \lambda_i$, by construction, is singular, there will only be $n-1$ independent equations to solve for the n elements of each of the eigenvectors. Eigenvectors give a direction in n-space, but not a distance in n-space. Consider the case of $n = 2$. In this case $\mathbf{v}_i$ solves:

$$\begin{bmatrix} a_{11}-\lambda_i & a_{12} \\ a_{21} & a_{22}-\lambda_i \end{bmatrix}\begin{bmatrix} v_{i1} \\ v_{i2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This expression consists of two linear equations, only one of which is independent. Arbitrarily choose the first of these and we find that:

$$(a_{11}-\lambda_i)v_{i1} + a_{12}v_{i2} = 0, \text{ and so:}$$

$$\frac{v_{i1}}{v_{i2}} = \frac{-a_{12}}{(a_{11}-\lambda_i)}$$

A convenient normalization is to set $v_{i1}^2 + v_{i2}^2 = 1$.

Systems of Differential Equations

Facts from linear algebra are helpful in solving systems of first-order linear, homogeneous differential equations. Such systems take the form $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ represent an $n \times n$ matrix, and $\dot{\mathbf{x}}$ and $\mathbf{x}$ represent $n \times 1$ vectors. Before engaging in a study of these more general cases, let us first turn our attention to the case of the homogeneous second-order differential equation with constant coefficients. That is, consider:

$$\ddot{y} + a\dot{y} + by = 0$$

Now consider a change in variables in which we define $x_1 = \dot{y}$ and $x_2 = y$. Therefore $\ddot{y} = \dot{x}_1$, $\dot{y} = x_1$, and $y = x_2$. We may therefore rewrite the original equation as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -a & -b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Let us guess a solution to the original differential equation that takes the form:

$$y(t) = ce^{\lambda t}$$

Now take the derivatives of this solution, $\dot{y} = c\lambda e^{\lambda t}$ and $\ddot{y} = c\lambda^2 e^{\lambda t}$. Now plug these expressions back into the original differential equation to obtain:

$$c\lambda^2 e^{\lambda t} + ac\lambda e^{\lambda t} + bce^{\lambda t} = (\lambda^2 + a\lambda + b)ce^{\lambda t} = 0,$$

if and only if $(\lambda^2 + a\lambda + b) = 0$, for $c \neq 0$, $e^{\lambda t} \neq 0$.

The solutions to the polynomial equation, $(\lambda^2 + a\lambda + b) = 0$, are identical to the eigenvalues of the matrix **A**. That is:

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} -a-\lambda & -b \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + a\lambda + b = 0$$

In this, a 2×2 case, there are two eigenvalues, λ_1 and λ_2. The solution therefore takes the form:

$$\begin{bmatrix} \dot{y} \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_{11}e^{\lambda_1 t} + c_{12}e^{\lambda_2 t} \\ c_{21}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t} \end{bmatrix}$$

We are primarily interested in the solution for $y(t)$, which is given by:

$$y(t) = c_{21}e^{\lambda_1 t} + c_{22}e^{\lambda_2 t}$$

The constants c_{21} and c_{22} are determined from the initial conditions. Once we find c_{21} and c_{22}, it is straightforward to solve for c_{11} and c_{12}.

Let us next consider how to deal with solutions to the generalized homogeneous system of two differential equations.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{x}.$$

So far, we are dealing with 2×2 systems, but the techniques you are learning here generalize to systems in higher dimensions. Systems of differential equations come up all the time in economics; the classic example in macroeconomic theory is the model of optimal growth in which both consumption and capital stock are determined simultaneously by the savings decision of the representative household. Thus this discussion is not an arid exercise in linear algebra, differential equations, and complex variables. Instead it constitutes the bread and butter of a lot of modern macroeconomics.

It should not be not surprising to you now that the cases of interest are essentially a taxonomy of the characteristic equation $p(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{21}a_{12}$. There are four cases: (1) $\lambda_1 = \lambda_2 = 0$; (2) both roots are real; (3) the two roots are complex conjugates; and (4) $\lambda_1 = \lambda_2 \neq 0$, in which the roots are the same real number.

Let's dispose of the first case immediately. If $\lambda_1 = \lambda_2 = 0$, then either $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or $\mathbf{A} = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}$ or $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}$. If any of these is true, then we are not dealing with a *system* of differential equations because either one or both variables is stationary and no variable's rate of change depends upon its own value.

If $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $\mathbf{x}(t) = \mathbf{x}(0)$.

If $\mathbf{A} = \begin{bmatrix} 0 & a_{12} \\ 0 & 0 \end{bmatrix}$, then $\mathbf{x}(t) = \begin{bmatrix} x_1(0) + a_{12}x_2(0)t \\ x_2(0) \end{bmatrix}$.

If $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ a_{21} & 0 \end{bmatrix}$, then $\mathbf{x}(t) = \begin{bmatrix} x_1(0) \\ x_2(0) + a_{21}x_1(0)t \end{bmatrix}$,
where $\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$ is given.

The second case is not so trivial. Let $\lambda_1 \neq \lambda_2$ both be real. Let $\mathbf{M}$ be the matrix whose *columns* are the two eigenvectors for $\lambda = \lambda_1$, and $\lambda = \lambda_2$. Since $\mathbf{AM} = \mathbf{M\Lambda} = \mathbf{M}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, we know that $\mathbf{\Lambda} = \mathbf{M}^{-1}\mathbf{AM}$. As long as $\mathbf{M}$ is non-singular, $\mathbf{M}^{-1}$ exists. For distinct eigenvalues, $(\lambda_1 \neq \lambda_2)$, $\mathbf{M}$ is always non-singular.

Now we do a nifty change of coordinates. Write $\mathbf{x} = \mathbf{My}$ and note that $\mathbf{y} = \mathbf{M}^{-1}\mathbf{x}$. Then $\dot{\mathbf{x}} = \mathbf{M}\dot{\mathbf{y}}$ since the $\mathbf{M}$ matrix is a matrix of constants. Hence we can rewrite our system as $\mathbf{M}\dot{\mathbf{y}} = \dot{\mathbf{x}} = \mathbf{Ax} = \mathbf{AMy}$ and thus

$$\dot{\mathbf{y}} = \mathbf{M}^{-1}\mathbf{AMy} = \Lambda\mathbf{y}.$$

But the solution to this equation is just $\mathbf{y}(t) = \begin{bmatrix} y_1(0)\exp(\lambda_1 t) \\ y_2(0)\exp(\lambda_2 t) \end{bmatrix}$. We are only halfway there because this solution is in the wrong basis; we have to get rid of the change of coordinates. In particular, we typically know $\mathbf{x}(0)$, not $\mathbf{y}(0)$. Recalling that $\mathbf{x} = \mathbf{M}\mathbf{y}$, we can write $\mathbf{y}(0) = \mathbf{M}^{-1}\mathbf{x}(0)$. So that pins down the initial condition in the changed coordinates. But also $\mathbf{x}(t) = \mathbf{M}\mathbf{y}(t) = \mathbf{M}\begin{bmatrix} y_1(0)\exp(\lambda_1 t) \\ y_2(0)\exp(\lambda_2 t) \end{bmatrix}$, and that is our answer.

To make things concrete, always follow these exact steps. First, calculate the eigenvalues and some corresponding eigenvectors. Second write the matrix whose columns are the eigenvectors and invert it. Third, use this inverse and your knowledge about the initial conditions to figure out $\mathbf{y}(0)$. Fourth, use $\mathbf{y}(0)$ and the matrix whose columns are the eigenvectors to get your answer in the original basis.

Here is an example. Let $\mathbf{A} = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\lambda_1 = 2$ and $\lambda_2 = -1$ are eigenvalues, and the matrix whose columns are the corresponding eigenvectors is $\mathbf{M} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$. The inverse of this matrix is $\mathbf{M}^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. So the initial condition in the changed coordinates is $\mathbf{y}(0) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Hence the solution in the changed basis is $\mathbf{y}(t) = \begin{bmatrix} 3\exp(2t) \\ 2\exp(-t) \end{bmatrix}$. Finally, the solution to our original problem is

$$\mathbf{x}(t) = \mathbf{M}\mathbf{y}(t) = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3\exp(2t) \\ 2\exp(-t) \end{bmatrix} = \begin{bmatrix} 3\exp(2t) - 2\exp(-t) \\ -3\exp(2t) + 4\exp(-t) \end{bmatrix}.$$

Check that this is the right answer.

We may also write a simple set of Matlab commands to perform the diagonalization procedure. This procedure also works for higher-order linear systems. The set of commands is given below.

```
A=[..]: Square matrix

[M,lambda]=eig(A)
```

t=sym('t')

z=exp(diag(lambda)*t)

x0=[..]: column vector of initial conditions

y0=inv(M)*x0

y=y0.*z

x=M*y

The third case occurs when the two roots are complex conjugates. The standard diagonalization procedure still works. The eigenvectors also occur in conjugate pairs. Much of the solution process involves working with complex vectors and matrices. Operations like computing the inverse of a matrix work with complex values as well as simple real numbers. However, when the process concludes, we obtain a real valued time function for the solution. Unfortunately, this procedure can give results which are not easy to interpret. To see this, try using the suggested Matlab technique for solving a 2×2 system with complex roots. It may appear that the time functions in the solution are complex. In fact, this is not the case.

An alternate solution technique separates out the real and imaginary parts of the eigenvalues and eigenvectors and performs an essentially idetical set of steps. For complex conjugate eigenvalues, can write $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ with $b > 0$. We are going to do a very similar process now, but we shall endeavour to write $\mathbf{A} = \mathbf{M}\mathbf{T}_{(a,b)}\mathbf{M}^{-1}$ where $\mathbf{T}_{(a,b)} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is the canonical (anti-clockwise) "twisting" matrix. Of course, if we find a clever change of coordinates and write $x = \mathbf{M}y$, then $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ will be equivalent to $\dot{\mathbf{y}} = \mathbf{M}^{-1}\mathbf{A}\mathbf{M}\mathbf{y} = \mathbf{T}_{(a,b)}y$.

How do we find the **M** matrix? Just solve for a complex-valued eigenvector that corresponds to the eigenvalue $\lambda_1 = a + bi$. Such an eigenvector can always be written in the form $\mathbf{z} = i\begin{bmatrix} m_{11} \\ m_{21} \end{bmatrix} + \begin{bmatrix} m_{12} \\ m_{22} \end{bmatrix}$ and then the matrix we are looking for is $\mathbf{M} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. Pay attention to the order of the columns. The reason that we put the imaginary part in the first column is that the definition of any eigenvector is $\mathbf{A}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 z_1 \\ \lambda_1 z_2 \end{bmatrix}$ where all these numbers can be complex. Since these are complex numbers. this equation is equivalent to:

$$\mathbf{A}\begin{bmatrix} im_{11} & m_{12} \\ im_{12} & m_{22} \end{bmatrix} = \begin{bmatrix} im_{11} & m_{12} \\ im_{12} & m_{22} \end{bmatrix}\mathbf{T}_{(a,b)},$$

where the first column on each side keeps track of the imaginary part and the second column keeps track of the real part. Thus $\mathbf{T}_{(a,b)} = \mathbf{M}^{-1}\mathbf{AM}$. The matrix $\mathbf{M}$ will have an inverse because the imaginary roots of the characteristic equation come in pairs. So this change of coordinates allows us to write the original equation $\dot{\mathbf{x}} = \mathbf{Ax}$ as $\dot{\mathbf{y}} = \mathbf{M}^{-1}\dot{\mathbf{x}} = \mathbf{M}^{-1}\mathbf{Ax} = \mathbf{M}^{-1}\mathbf{AMy} = \mathbf{T}_{(a,b)}\mathbf{y}$. From our knowledge about complex variables, we know that:

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \exp(at)\begin{bmatrix} u\cos tb - v\sin tb \\ u\sin tb + v\cos tb \end{bmatrix},$$

for given initial conditions $\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$.

Again, we can always compute $\mathbf{y}(0) = \mathbf{M}^{-1}\mathbf{x}(0)$ because $\mathbf{x}(0)$ is what we were really given. Then

$$\mathbf{x}(t) = \mathbf{M}\begin{bmatrix} \exp(at)(y_1(0)\cos tb - y_2(0)\sin tb) \\ \exp(at)(y_1(0)\sin tb + y_2(0)\cos tb) \end{bmatrix}$$

is the answer to the initial value problem.

Here is another example. Let $\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ are the eigenvalues. So we already know that our canonical twisting matrix will be $\mathbf{T}_{(a,b)} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Now we need to compute an eigenvector corresponding to $\lambda_1 = 1 + i$. (Always pick the one with the positive imaginary part.) Using the definition of an eigenvector, we have

$$(\mathbf{A} - \lambda_1\mathbf{I})z = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(It may not be obvious, but the top and bottom rows give the same information; you can check this by computing $2/(-1-i)$.) The bottom row tells us that an easy eigenvector is:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} (1-i) \\ -1 \end{bmatrix} = i\begin{bmatrix} -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then the matrix giving the change of coordinates is **M** $= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$. The inverse of this matrix is $\mathbf{M}^{-1} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$. This means that we wan rewrite $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ as:

$$\dot{\mathbf{y}} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\mathbf{y} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{y}.$$

Also, the initial condition in the changed coordinates is:

$$\mathbf{y}(0) = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}.$$

The solution in the changed basis is:

$$\mathbf{y}(t) = \exp(t)\begin{bmatrix} -2\cos t + \sin t \\ -2\sin t - \cos t \end{bmatrix}.$$

Finally, the solution to our original problem is

$$\mathbf{x}(t) = \mathbf{M}\mathbf{y}(t) = \exp(t)\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -2\cos t + \sin t \\ -2\sin t - \cos t \end{bmatrix}$$

$$= \exp(t)\begin{bmatrix} \cos t - 3\sin t \\ \cos t + 2\sin t \end{bmatrix}.$$

Again, check that this is the right answer.

We may also write a somewhat complicated Matlab routine which follows these same steps. Try the procedure outlined as follows:

```
A=[ ]
x0=[ ]
[M lambda]=eig(A)
[C I]=max(imag(lambda))
b=M(:,I(1))
b2=real(b)
b1=imag(b)
M=[b1';b2']'
y0=inv(M)*x0
a=real(lambda(1,1))
b=max(max(imag(lambda)))
y1=sym('y1')
y2=sym('y2')
```

```
t=sym('t')
u=y0(1)
v=y0(2)
y1=exp(a*t)*(u*cos(b*t)-v*sin(b*t))
y2=exp(a*t)*(v*cos(b*t)+u*sin(b*t))
y=[y1;y2]
x=M*y
x=simplify(x)
```

The fourth and final case occurs when **A** has two identical (real) roots that are not zero. Write $\lambda = \lambda_1 = \lambda_2 \neq 0$. There are two possibilities. First, if $\mathbf{A} = \lambda\mathbf{I}$, then the differential equation is already uncoupled, and we can solve it immediately. If $\mathbf{A} \neq \lambda\mathbf{I}$, then you can write $\mathbf{N} = \mathbf{A} - \lambda\mathbf{I}$ where $\mathbf{N}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the general solution to $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can be written as $\mathbf{x}(t) = \exp(\lambda t)(\mathbf{I} + t\mathbf{N})$, and the definite solution is $\mathbf{x}(t) = \exp(\lambda t)(\mathbf{I} + t\mathbf{N})\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$. (Explaining why these facts are true is beyond the scope of this course. They are based on a Taylor series expansion for operators on matrices.)

Our final example will be $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$ with $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The characteristic equation has $(1-\lambda)(3-\lambda) + 1 = 0$, and thus $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$. So there are two identical roots. Now we write $\mathbf{N} = \mathbf{A} - 2\mathbf{I} = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. (Make sure that $\mathbf{N}^2 = 0$.) We may conclude that:

$$\mathbf{x}(t) = \exp(2t)(\mathbf{I} + t\mathbf{N})\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \exp(2t)\begin{bmatrix} 1 - 2t \\ 1 + 2t \end{bmatrix}.$$

Finally, check that this is the right answer.