Difference Equations

Variables at Discrete Intervals

Example:

$$y_t \,:\, \text{Value of } y \text{ at time } t$$

$$y_{t+n} - y_t = n$$

$$n \,:\, \text{a given constant}$$

Solution is a function of time that conforms to the equation.

For this example, try:

$$y_t = t$$

In that case, $y_{t+n} = t + n$, and so:

$$y_{t+n} - y_t = t + n - t = n$$

Therefore, $y_t = t$ is a solution to the difference equation.

Alternatively, for any constant, k, consider:

$$y_t = t + k$$

Now, $y_{t+n} = t + k + n$, and so:

$$y_{t+n} - y_t = t + k + n - (t + k) = n$$

Solution is not unique.

There is one solution for every possible value of k.

Solution is unique if we also require that, for example, $y_1 = 3$.

For $y_t = t + k$, and for $y_1 = 3$, we require that $k = 2$:

The unique solution is given by:

$$y_t = t + 2$$

Linear, first-order, homogeneous, constant coefficient

$y_{t+1} = ay_t$, $a =$ constant.

Guess the solution:

$$y_t = ca^t$$

Check: $y_{t+1} = ca^{t+1}$

$$y_{t+1} - ay_t = ca^{t+1} - aca^t = 0$$

Initial conditions pins down constant, c.

Linear, first-order, non-homogeneous, constant coefficient

$$y_{t+1} - ay_t = q(t)$$

Solution consists of two parts

Complementary Solution: $y_t = ca^t$

Particular Solution: Must be found case by case

Suppose that we have

$$y_{t+1} - ay_t = B$$

Guess the particular solution $y_{pt} = C$.

Therefore $y_{pt+1} = y_{pt} = C$, and so:

$$C - aC = B \Rightarrow C = \frac{B}{1-a}$$

Finally:

$$y_t = y_{ct} + y_{pt} = ca^t + \frac{B}{1-a}$$

Linear, second-order, homogeneous, constant coefficient

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0$$

Guess the solution:

$$y_t = c\lambda^t$$

Check: $y_{t+1} = c\lambda^{t+1}$, $y_{t+2} = c\lambda^{t+2}$

Plug in: $c\lambda^{t+2} + a_1 c\lambda^{t+1} + a_2 c\lambda^t = 0$

$$c\lambda^t[\lambda^2 + a_1\lambda + a_2] = 0$$

Therefore:

$$\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2} \Rightarrow \lambda_1, \lambda_2$$

Finally:

$$y_t = c_1\lambda_1^t + c_2\lambda_2^t$$

Unique solution for two intial conditions.

Three subcases:

λ_1, λ_2 real and distinct

λ_1, λ_2 real and identical, $\lambda_1 = \lambda_2 = \lambda$

λ_1, λ_2 complex reciprocal pair.

Identical Root Solution:

$$y_t = c_1\lambda^t + c_2 t\lambda^t$$

Complex Root Solution:

$$\begin{aligned} \lambda_1 &= g + ih \\ \lambda_2 &= g - ih \\ y_t &= \rho^t[c_1 \cos\theta t + c_2 \sin\theta t] \\ \rho &= \sqrt{g^2 + h^2} \\ \theta &= \tan^{-1}\left(\frac{h}{g}\right) \end{aligned}$$

Lag Operators and Non-homogeneous Equations

Define the operator, L, such that:

$Lx_t = x_{t-1}$, and $\frac{1}{L}x_t = x_{t+1}$

L is a linear operator: standard arithmetic operations are valid.

$(1 + L)x_t = x_t + x_{t-1}$, $L^2 x_t = x_{t-2}$, etc.

Lag operator quite helpful with non-homogeneous difference equations.

Example: $x_{t+1} = \lambda x_t + w_{t+1}$, or: $x_t = \lambda x_{t-1} + w_t$

Rewrite: $x_t = \lambda L x_t + w_t$

$$(1 - \lambda L)x_t = w_t$$

$$x_t = \frac{1}{(1 - \lambda L)} w_t + c\lambda^t$$

The $c\lambda^t$ term must be included because that is the complementary solution to the difference equation. To check that this term is sensibly included, multiply by $(1 - \lambda L)$ to obtain:

$$\begin{aligned} (1 - \lambda L)x_t &= w_t + (1 - \lambda L)c\lambda^t \\ &= w_t + c[\lambda^t - \lambda L \lambda^t] \\ &= w_t + c[\lambda^t - \lambda\lambda^{t-1}] \\ &= w_t \ \forall c \end{aligned}$$

Now consider the term, $\frac{1}{(1-\lambda L)}$.

Rewrite $\frac{1}{(1 - \lambda L)} = 1 + \lambda L + \lambda^2 L^2 + \lambda^3 L^3 + \cdots$

Alternatively

$$\begin{aligned} \frac{1}{(1 - \lambda L)} &= \frac{-(\lambda L)^{-1}}{[1 - (\lambda L)^{-1}]} \\ &= -\frac{1}{\lambda L}\left(1 + \frac{1}{\lambda}L^{-1} + \frac{1}{\lambda^2}L^{-2} + \cdots\right) \end{aligned}$$

First series converges for $|\lambda| < 1$, second converges for $|\lambda| > 1$

For $|\lambda| < 1$: $x_t = w_t + \lambda w_{t-1} + \lambda^2 w_{t-2} + \cdots + c\lambda^t$

System running forever: $x_t = \sum_{i=0}^{\infty} \lambda^i u_{t-i} + c\lambda^t$

System with start date, "0": $x_t = \lambda^t x_0 + \sum_{i=0}^{t-1} \lambda^i w_{t-i}$

$$\begin{aligned} \text{For } |\lambda| > 1\text{: } x_t &= -(\lambda L)^{-1}\left[w_t + \frac{1}{\lambda} w_{t+1} + \frac{1}{\lambda^2} w_{t+2} + \cdots \right] + c\lambda^t \\ &= -\frac{1}{\lambda} w_{t+1} - \frac{1}{\lambda^2} w_{t+2} - \frac{1}{\lambda^3} w_{t+3} - \cdots + c\lambda^t \\ &= \sum_{j=1}^{\infty} \left(\frac{1}{\lambda} \right)^j w_{t+j} + c\lambda^t \end{aligned}$$

According to whether $|\lambda| \gtrless 1$, equation is solved "forward" or "backwards."

Systems of Difference Equations

Consider the vector - matrix difference equation:

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t$$

$\mathbf{x}$: $n \times 1$ vector of state variables

$\mathbf{A}$: $n \times n$ matrix of constants.

$$\begin{bmatrix} x_{1t+1} \\ x_{2t+1} \\ \vdots \\ x_{nt+1} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_{1t} \\ x_{2t} \\ \vdots \\ x_{nt} \end{bmatrix}$$

For now, consider the homogeneous case.

Eigenvalues and Eigenvectors

Consider a square matrix, $\mathbf{A}$

Eigenvalues, λ_i are solutions to:

$$|\mathbf{A}-\lambda\mathbf{I}| = \left|\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}\right| = 0$$

Eigenvalues turn out to be solutions to the extended characteristic equation.

Some useful properties of Eigenvalues:

$\lambda_1\lambda_2\cdots\lambda_n = |\mathbf{A}|$ and $\lambda_1+\lambda_2+\ldots+\lambda_n = trace(\mathbf{A})$

Matlab command eig(A) gives eigenvalues of **A**

Eigenvectors

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i, or: (\mathbf{A}-\lambda_i\mathbf{I})\mathbf{v}_i = \mathbf{0}$$

$\mathbf{v}_i$'s, **0** are column vectors of length n.

Definitions:

M: Modal matrix; columns are eigenvectors

$$\mathbf{M} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}$$

Λ: Diagonal matrix of eigenvalues

$$\Lambda \equiv \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Matlab command [M,lambda]=eig(A):

'M': modal matrix of **A**

'lambda': Diagonal matrix of eigenvalues of **A**

Diagonalization

Define: **M**: modal matrix of **A**

Λ: diagonal matrix of eigenvalues of **A**

For each eigenvalue:

$$\mathbf{A}\mathbf{v}_i = \lambda_i\mathbf{v}_i$$

Therefore:

$$\begin{bmatrix} \mathbf{A}\mathbf{v}_1 & \mathbf{A}\mathbf{v}_2 & \cdots & \mathbf{A}\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\mathbf{v}_1 & \lambda_2\mathbf{v}_2 & \cdots & \lambda_n\mathbf{v}_n \end{bmatrix}$$

$$\mathbf{A}\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

Equivalently:

$$\mathbf{AM} = \mathbf{M}\Lambda$$

If **M** is non-singular, $\mathbf{M}^{-1}$ exists

M is non-singular when λ's are unique (not repeated).

Premultiply by $\mathbf{M}^{-1}$:

$$\Lambda = \mathbf{M}^{-1}\mathbf{AM}$$

Also:

$$\mathbf{A} = \mathbf{M}\Lambda\mathbf{M}^{-1}$$

Transformed variable: $\mathbf{z} \equiv \mathbf{M}^{-1}\mathbf{x}$, and so: $\mathbf{x} = \mathbf{Mz}$

$$\mathbf{z}_{t+1} = \mathbf{M}^{-1}\mathbf{x}_{t+1} = \mathbf{M}^{-1}\mathbf{A}\mathbf{x}_t = \mathbf{M}^{-1}\mathbf{AM}\mathbf{z}_t$$

$$\mathbf{z}_{t+1} = \Lambda\mathbf{z}_t$$

Solution:

$$\begin{bmatrix} z_{1t} \\ z_{2t} \\ \vdots \\ z_{nt} \end{bmatrix} = \begin{bmatrix} \lambda_1^t z_{10} \\ \lambda_2^t z_{20} \\ \vdots \\ \lambda_n^t z_{n0} \end{bmatrix} = \Lambda^t \mathbf{z}_0$$

where $\mathbf{z}_0 = \mathbf{M}^{-1}\mathbf{x}_0$, and finally:

$$\mathbf{x}_t = \mathbf{M}\mathbf{z}_t = \mathbf{M}\Lambda^t\mathbf{M}^{-1}\mathbf{x}_0$$

Stability: $\lim_{t\to\infty} \mathbf{x}_t = \lim_{t\to\infty} \mathbf{M}\Lambda^t\mathbf{M}^{-1}\mathbf{x}_0 = \mathbf{0}\ (n \times 1)$

If all of the eigenvalues of **A** lie within the unit circle, $\mathbf{x}_t$ is stable.

Matlab Implementation

```
t=sym('t')
A=[  ]
```

```
x0=[  ]
[M,L]=eig(A)
x=M*L.^t*inv(M)*x0
To Evaluate: xn=subs(x,t,n)
e.g., x3=subs(x,t,3)
```

Direct Solution

Direct Solution: $\mathbf{x}_t = \mathbf{A}^t\mathbf{x}_0$

Matrix exponentiation: $\mathbf{A}^t \equiv \mathbf{A} \cdot \mathbf{A} \cdot \cdots \cdot \mathbf{A}$, "$t$" times.

Stability requires $\lim_{t\to\infty} \mathbf{A}^t = \mathbf{0}$ $(n \times n)$

From above, $\lim_{t\to\infty} \mathbf{A}^t = \mathbf{0}$ if all of the eigenvalues of **A** lie within the unit circle

Non-homogeneous Case

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{C}\mathbf{w}_{t+1}$$

w is an $m \times 1$ vector

C is an $n \times m$ matrix.

Solution

Lag operator can also be applied to vectors.

$$(\mathbf{I} - \mathbf{A}L)\mathbf{x}_{t+1} = \mathbf{C}\mathbf{w}_{t+1}$$

Equivalently, $(\mathbf{I} - \mathbf{A}L)\mathbf{x}_t = \mathbf{C}\mathbf{w}_t$, and so:

$$\mathbf{x}_t = \mathbf{C}\mathbf{w}_t + \mathbf{A}\mathbf{C}\mathbf{w}_{t-1} + \mathbf{A}^2\mathbf{C}\mathbf{w}_{t-2} + \cdots + \mathbf{A}^t\mathbf{k}$$

$\mathbf{x}_t = \sum_{j=0}^{\infty} \mathbf{A}^j\mathbf{C}\mathbf{w}_{t-j} + \mathbf{A}^t\mathbf{k}$, as long as all of the eigenvalues of **A** lie within the unit circle

In this equation, **k** is an $n \times 1$ vector of constants and $\mathbf{A}^t\mathbf{k}$ is the complementary solution of the system of difference equations. Note that $(\mathbf{I} - \mathbf{A}L)\mathbf{A}^t\mathbf{k} = \mathbf{0}, \forall \mathbf{k}$. If we are given an inital start date and an initial condition, $\mathbf{x}(0) = \mathbf{x}_0$, then the solution can be written as:

$$\mathbf{x}_t = \mathbf{A}^t\mathbf{x}_0 + \sum_{j=0}^{t-1} \mathbf{A}^j\mathbf{C}\mathbf{w}_{t-j}$$

Stochastic Linear Difference Equations

$$x_{t+1} = Ax_t + Cw_{t+1}$$

Suppose that the $w's$ are random variables.

Simplest Case:

$$\begin{aligned} E\mathbf{w}_{t+1}|J_t &= 0 \\ E\mathbf{w}_{t+1}\mathbf{w}'_{t+1}|J_t &= \mathbf{I} \\ J_t &: \text{ Period } t \text{ information set} \\ J_t &= [\mathbf{w}_t, \mathbf{w}_{t-1}, \cdots, \mathbf{w}_1, \mathbf{x}_0] \end{aligned}$$

For non-zero mean, add a constant.
For non-unity variance, multiply by a constant.
These properties also imply:

$$E_t[\mathbf{w}_{t+i}\mathbf{w}'_{t+j}] = \mathbf{0},\ i > 0,\ j > 0,\ i \neq j$$

Covariance Stationary Stochastic Processes

Denote unconditional means as $E\{\quad\}$.
The process, $\{x_t\}_{t=0}^{\infty}$, is covariance stationary if:

$$E\mathbf{x}_t = E\mathbf{x}_0,\ \forall t, \text{ and}$$

$$E\{(\mathbf{x}_{t+j} - E\mathbf{x}_{t+j})(\mathbf{x}_t - E\mathbf{x}_t)'\}, \text{ is independent of } t$$

The process must have a constant unconditional mean.
Covariances depend on the separation between dates, j.
Covariances are independent of calander time, t.
Consider the process:

$$\mathbf{x}_{t+1} = \mathbf{A}\mathbf{x}_t + \mathbf{C}\mathbf{w}_{t+1}$$

Denote the mean of x_t as μ_t.

$$E(\mathbf{x}_{t+1}) \equiv \boldsymbol{\mu}_{t+1} = E(\mathbf{A}\mathbf{x}_t + \mathbf{C}\mathbf{w}_{t+1}) = \mathbf{A}\boldsymbol{\mu}_t$$

Is this process covariance stationary?
Typically, $\mu_t = 0$, unless one of x_{it} is constant.
If one of the state variables,x_{it}, is a constant, then **A** will have one eigenvalue of unity.
Suppose that this is the case.

$$\boldsymbol{\mu}_{t+1} = \mathbf{A}\boldsymbol{\mu}_t$$

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\mu}_t = \boldsymbol{\mu} \text{ if}$$

$$(\mathbf{I} - \mathbf{A})\boldsymbol{\mu} = \mathbf{0}$$

μ is the eigenvector associated with the unit eigenvalue.
Recall the solution:

$$\mathbf{x}_t = \mathbf{A}^t\mathbf{x}_0 + \sum_{j=0}^{t-1}\mathbf{A}^j\mathbf{C}\mathbf{w}_{t-j}$$

For any $\mathbf{x}_0$, $E_0\mathbf{x}_t$ must approach **μ** as $t \to +\infty$.

Therefore $\lim_{t\to+\infty} \mathbf{A}^t =$ bounded constant.

The other eigenvalues of **A** must be less than unity in modulus.

Variances and Covariances

Define:

$$\mathbf{C}_x(0) = E\{(\mathbf{x}_t - E\mathbf{x}_t)(\mathbf{x}_t - E\mathbf{x}_t)'\}$$

Stationarity implies that $E\mathbf{x}_t = \boldsymbol{\mu}_t = \boldsymbol{\mu}$, and so:

$$\begin{aligned} E\{(\mathbf{x}_t - E\mathbf{x}_t)(\mathbf{x}_t - E\mathbf{x}_t)'\} &= E\{(\mathbf{x}_{t+1} - E\mathbf{x}_{t+1})(\mathbf{x}_{t+1} - E\mathbf{x}_{t+1})'\} \\ &= E\{(\mathbf{x}_{t+1} - \boldsymbol{\mu})(\mathbf{x}_{t+1} - \boldsymbol{\mu})'\} \end{aligned}$$

where:

$$\mathbf{x}_{t+1} - \boldsymbol{\mu} = \mathbf{A}\mathbf{x}_t + \mathbf{C}\mathbf{w}_{t+1} - \boldsymbol{\mu}$$

We also know that $\boldsymbol{\mu} = \mathbf{A}\boldsymbol{\mu}$:

$$(\mathbf{x}_{t+1} - \boldsymbol{\mu}) = \mathbf{A}\mathbf{x}_t + \mathbf{C}\mathbf{w}_{t+1} - \boldsymbol{\mu} = \mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}) + \mathbf{C}\mathbf{w}_{t+1}$$

Now compute the righthand-side of the $\mathbf{C}_x(0)$ equation:

$$\begin{aligned} \mathbf{C}_x(0) &= E((\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}) + \mathbf{C}\mathbf{w}_{t+1})(\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}) + \mathbf{C}\mathbf{w}_{t+1})') \\ &= E\{(\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}) + \mathbf{C}\mathbf{w}_{t+1})([\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu})]' + [\mathbf{C}\mathbf{w}_{t+1}]')\} \\ &= E\{(\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu}) + \mathbf{C}\mathbf{w}_{t+1})((\mathbf{x}_t - \boldsymbol{\mu})'\mathbf{A}' + \mathbf{w}_{t+1}'\mathbf{C}')\} \\ &= E\{\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'\mathbf{A}' \\ &\quad + \mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu})\mathbf{w}_{t+1}'\mathbf{C}' \\ &\quad + \mathbf{C}\mathbf{w}_{t+1}(\mathbf{x}_t - \boldsymbol{\mu})'\mathbf{A}' \\ &\quad + \mathbf{C}\mathbf{w}_{t+1}\mathbf{w}_{t+1}'\mathbf{C}'\} \end{aligned}$$

Therefore:

$$\begin{aligned} \mathbf{C}_x(0) &= E\{\mathbf{A}(\mathbf{x}_t - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'\mathbf{A}'\} + \mathbf{C}\mathbf{I}\mathbf{C}' \\ &= \mathbf{A}E\{(\mathbf{x}_t - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'\}\mathbf{A}' + \mathbf{C}\mathbf{I}\mathbf{C}' \\ &= \mathbf{A}\mathbf{C}_x(0)\mathbf{A}' + \mathbf{C}\mathbf{C}' \end{aligned}$$

This is a discrete Liaponov equation.

Typically solved with numerical methods, e.g., Sargent’s doublej.m

We may now solve for the general autocovariance term:

$$\mathbf{C}_x(j) = E\{(\mathbf{x}_{t+j} - \boldsymbol{\mu})(\mathbf{x}_t - \boldsymbol{\mu})'\}$$

Since $\boldsymbol{\mu} = \mathbf{A}\boldsymbol{\mu}$, then $\boldsymbol{\mu} = \mathbf{A}(\mathbf{A}\boldsymbol{\mu}) = \mathbf{A}^2\boldsymbol{\mu}$, and for any $j, \boldsymbol{\mu} = \mathbf{A}^j\boldsymbol{\mu}$ Rewrite:

$$\mathbf{x}_{t+j} - \boldsymbol{\mu} = \mathbf{A}^j(\mathbf{x}_t - \boldsymbol{\mu}) + \mathbf{C}\mathbf{w}_{t+j} + \mathbf{A}\mathbf{C}\mathbf{w}_{t+j-1} + \cdots \mathbf{A}^{j-1}\mathbf{C}\mathbf{w}_{t+j}$$

Postmultiply by $(\mathbf{x}_t - \boldsymbol{\mu})'$:

$$
\begin{aligned}
(\mathbf{x}_{t+j}-\boldsymbol{\mu})(\mathbf{x}_t-\boldsymbol{\mu})' &= \mathbf{A}^j(\mathbf{x}_t-\boldsymbol{\mu})(\mathbf{x}_t-\boldsymbol{\mu})' \\
&+ \mathbf{C}\mathbf{w}_{t+j}(\mathbf{x}_t-\boldsymbol{\mu})' \\
&+ \mathbf{A}\mathbf{C}\mathbf{w}_{t+j-1}(\mathbf{x}_t-\boldsymbol{\mu})' \\
&+ \cdots \\
&+ \mathbf{A}^{j-1}\mathbf{C}\mathbf{w}_{t+j}(\mathbf{x}_t-\boldsymbol{\mu})'
\end{aligned}
$$

Now take the expected value of this expression.
All but the first righthand side terms disappear.
Therefore:

$$\mathbf{C}_x(j) = \mathbf{A}^j\mathbf{C}_x(0)$$

Suppose that:

$$\mathbf{y}_t = \mathbf{G}\mathbf{x}_t$$

The variable, y_t, is a linear combination of the x_t’s.
Moments of y_t are easily computed as:

$$E\mathbf{y}_t = \mathbf{G}\boldsymbol{\mu} = \boldsymbol{\mu}_y$$

$$E\{(\mathbf{y}_{t+j}-\boldsymbol{\mu}_y)(\mathbf{y}_t-\boldsymbol{\mu}_y)'\} = \mathbf{G}\mathbf{C}_x(j)\mathbf{G}'$$

Suppose that the $w's$ are random variables.
Simplest Case:

$$
\begin{aligned}
E\mathbf{w}_{t+1}|J_t &= 0 \\
E\mathbf{w}_{t+1}\mathbf{w}'_{t+1}|J_t &= \mathbf{I} \\
J_t &: \text{ Period } t \text{ information set} \\
J_t &= [\mathbf{w}_t, \mathbf{w}_{t-1}, \cdots, \mathbf{w}_1, \mathbf{x}_0]
\end{aligned}
$$

For non-zero mean, add a constant.
For non-unity variance, multiply by a constant.
These properties also imply:

$$E_t[\mathbf{w}_{t+i}\mathbf{w}'_{t+j}] = \mathbf{0},\ i > 0,\ j > 0,\ i \neq j$$