

Introduction to Differential Equations

If we have a time function, $y(t)$, we may take its time derivative, $\dot{y} = dy/dt$. For example, if $y(t) = t^2$, then $\dot{y} = dy/dt = 2t$. With differential equations, we reverse the procedure. Therefore, if we know that $\dot{y} = 2t$, then we may wish to solve for $y(t)$. In this example, we therefore know that when $t = 1$, $dy/dt = 2$. However, there are an infinite number of functions, $y(t)$, for which $dy/dt = 2$, at $t = 1$. For example, for any constant, c , $y(t) = t^2 + c$ also solves $dy/dt = 2$. In this example, we also know that at $t = 2$, $dy/dt = 4$.

To pin down the correct solution, we need some additional information. We also need to specify one point on $y(t)$. In this example, we might know that $y(t) = t^2$ and that $y(2) = 4$. Therefore:

$$y(t) = t^2 + c \Rightarrow y(2) = t^2 + c = 2^2 + c = 4.$$
$$c = 0, \text{ and } y(t) = t^2$$

For the general case of $\dot{y} = f(y, t)$ we wish to know whether a solution, $y(t)$ exists and whether it is unique.

Theorem *Suppose that $f(\cdot)$ and its derivatives with respect to time are continuous in a region, R^2 . Let (y_0, t_0) be a point in R^2 . Then for each point, (y_0, t_0) : (a) there exists a solution of the differential equation, and (b) it is unique.*

This is known as the fundamental existence theorem. In our example, $f(y, t) = t^2$, is continuous and its derivatives are continuous. Therefore a solution exists and it is unique. The solution in this case is $y(t) = t^2$.

The central question is how to find the solution. In some cases, like this simple example, the solution will be straightforward. In other cases, we may guess the right solution. In any case, if we find a solution, we know that it is unique and therefore don't need to look for other solutions.

This first example is special in that it centers on a first-order differential equation. The *order* of a differential equation is given by the highest order derivative of y . In this case it is the first derivative, dy/dt . More complicated cases include higher order derivatives like d^2y/dt^2 , d^3y/dt^3 , etc. This case is also special in that there is only one independent variable, t . Such a differential equation is called an *ordinary differential equation*. Differential equations with more than one independent variable are called *partial differential equations*. For example, such equations may include the independent variables, height, width, and length.

A differential equation is a *linear* differential equation if it can be written in the form:

$$h_0(t) \frac{d^n y}{dt^n} + h_1(t) \frac{d^{n-1} y}{dt^{n-1}} + h_2(t) \frac{d^{n-2} y}{dt^{n-2}} + \dots + h_{n-1}(t) \frac{dy}{dt} + h_n(t) y(t) = f(t)$$

Such a linear differential equation is a constant coefficients linear differential equation if the coefficients, $h_0(t), h_1(t), \dots, h_n(t)$ are all time-constant.

Let us start our analysis with first-order differential equations. In these equations, only the first derivative matters. The canonical form is $\dot{y} \equiv dy/dt = f(t, y)$, and it has a solution under some fairly general conditions. Here are some special cases that are very useful for economists.

1. The simplest case is: $\dot{y} = f(t)$ with $y(t_0) = y_0$ given. Then the definite solution is

$$y(t) = y_0 + \int_{t_0}^t f(s) ds.$$

This corresponds to the relationship between flows and stocks in economics. It is often convenient to use the notation $F(t) = \int_{t_0}^t f(s) ds$. The argument t_0 has been suppressed here.

2. The next simplest case is: $\dot{y} + ay = 0$, with $y(t_0) = y_0$ given. Then $y(t) = y_0 \exp(-at)$. This corresponds to the notion of constant growth or discounting. An easy way to remember this solution is to recall that $dy/y = -adt$ can be integrated directly to give: $\ln(y) = c - at$, where c is a constant of integration. This differential equation is *homogenous* because $y(t) = y_0 = 0$ is always a solution, even though it may not be stable.

This simple case also introduces the notion of the stability of a solution. If $-a < 0$, then $\lim_{t \rightarrow \infty} y(t) = 0$ for any y_0 . This solution is *stable*. If $-a > 0$, then $\lim_{t \rightarrow \infty} y(t)$ exists only if $y_0 = 0$. This solution is *unstable*. If $a = 0$ then $\lim_{t \rightarrow \infty} y(t) = y_0$ for any initial condition. Thus the original equation doesn't even describe a dynamic system!

3. The next case combines the first two. Now consider: $\dot{y} + ay = b$ with $a \neq 0$. The solution to this differential equation illustrates the notions of a *complementary* function and a *particular* integral. The complementary function is just the solution to the homogeneous equation. Thus $y_c = c \exp(-at)$ is the complementary function, where c is an arbitrary constant. The particular integral is *any* solution of the equation, including the case where the equation is a constant. Thus $y_p = b/a$ is the particular integral. Then the *general* solution is the sum of the complementary function and the particular integral $y(t) = c \exp(-at) + b/a$. If $y(t_0) = y_0$ is given, then the *definite* solution is: $y(t) = [y_0 - b/a] \exp(-at) + b/a$. It is often convenient to rewrite this equation as

$y(t) - b/a = [y_0 - b/a] \exp(-at)$. Thus in economics the quantity b/a can be thought of as the long-run equilibrium value for the system, and $a > 0$ is the speed of adjustment to the stable equilibrium.

4. What about a more general linear equation of the form $\dot{y} + a(t)y = b(t)$, with $y(t_0) = y_0$ given? We assume that $a(t)$ is an integrable function. The definition of an integrable function is fairly involved and it involves important aspects of measure theory. First, let $\mu(t) = \exp\left[\int_{t_0}^t a(s)ds\right]$. This expression is called the integrating factor.

Now multiply the original equation by $\mu(t)$ to get: $\mu(t)\dot{y} + \mu(t)a(t)y = \mu(t)b(t)$. The product rule for differentiation and the fundamental theorem of the calculus implies that $\frac{d}{dt}\mu(t)y = \mu(t)\dot{y} + \mu(t)a(t)y$. Now we can integrate both sides and derive that

$$\mu(t)y = \int_{t_0}^t \mu(s)b(s)ds + c \text{ where } c \text{ is an arbitrary constant. Thus } y(t) = \frac{\int_{t_0}^t \mu(s)b(s)ds + c}{\mu(t)},$$

and this expression is well defined since $\mu(t) \neq 0$. Using the initial condition, we see that $y_0 = c$, and we derive that $\mu(t)y(t) = y_0 + \int_{t_0}^t \mu(s)b(s)ds$. This expression has an

economic interpretation. Think of $\mu(t) = \exp\left[\int_{t_0}^t a(s)ds\right]$ as the cumulative interest factor for a time-varying interest rate $a(t)$. Then $\int_{t_0}^t \mu(s)b(s)ds$ is the future value of the flow described by $b(t)$. One simple interpretation would be that the present value of wealth is equal to initial wealth plus the present value of cumulative savings.

5. A related procedure, variation of parameters is a way to find the particular integral is called the variation of parameter approach. In this approach, we start by assuming that the particular integral is proportional to the complementary function. For the first-order linear case, the differential equation is of the form $\dot{y} + ay = b$ with $a \neq 0$. Suppose that a is constant, but that b is a function of time. The complementary function is of the form, $y_c = c \exp(-at)$. Now assume that $y_p(t) = u(t)y_c(t)$. Therefore $\dot{y}_p(t) = u\dot{y}_c + \dot{u}y_c$. Now evaluate $y_p(t)$ in the original equation:

$$\begin{aligned} \dot{y} + a(t)y &= u\dot{y}_c + \dot{u}y_c + au y_c = b(t) \\ &= u(\dot{y}_c + ay_c) + ay_c \dot{u} = y_c \dot{u} = b(t) \end{aligned}$$

We may now integrate $\frac{du}{dt} = \frac{b(t)}{ay_c(t)}$ to obtain:

$$u(t) = \int \frac{b(t)}{y_c(t)} dt = \int \frac{b(t)}{c \exp(-at)} dt$$

Finally,

$$y_p(t) = u(t)y_c(t) = c \exp(-at) \left(\int \frac{b(t)}{c \exp(-at)} dt \right)$$

As with all of these procedures, we will need some good fortune in obtaining an easily integrable functional form.

6. Let us consider non-linear differential equations that are separable in a special way. In particular, suppose that a differential equation can be written in the form

$$M(t) + N(y) \frac{dy}{dt} = 0. \text{ Then the general solution to this equation can be written}$$

$\int M(t)dt = -\int N(y)dy$. Now this may not yield a very good explicit solution for y , but it will be free of its differential forms. Here it is better to substitute any initial conditions first, because they may allow some variables to drop out, and make solving for an explicit solution simpler. A typical example in economics might involve a cyclical forcing term $M(t)$ and a convex adjustment cost term $-N(y)$.

7. We will finally consider the method of exact differential equations. Consider a function of two variables $F(y, t)$ and its total differential $dF(y, t) = (\partial F/\partial y)dy + (\partial F/\partial t)dt$. If $(\partial F/\partial y)dy + (\partial F/\partial t)dt = 0$, then this equation is an *exact differential equation*. The general form is $M(y, t)dy + N(y, t)dt = 0$. Since the cross-partial derivatives are equal, we see that the general form describes an exact differential equation only if $\partial M/\partial t = \partial^2 F/\partial y\partial t = \partial N/\partial y$. (In complex analysis, this equation is called the Cauchy-Riemann equation and it is a necessary condition for the original complex-valued function $F(y, t)$ to have a derivative.) This equation is quick test of the whether a differential equation does have a solution. If it does, then it is clear that the general solution will be of the form $F(y, t) = c$ for some arbitrary constant c .

So how does one solve an exact differential equation? There are four steps. First, write $F(y, t) = \int Mdy + \psi(t)$ where the integral is with respect to the partial derivative $M(y, t)$ and t is treated as a constant. The function $\psi(t)$ shows that the solution will depend in general upon time. Second, differentiate the solution from the first step with respect to t and derive $\partial F/\partial t = N(y, t) = \frac{\partial}{\partial t} \int Mdy + \dot{\psi}(t)$. This yields a solution for $\dot{\psi}(t)$ since $N(y, t)$ is known. Third, integrate the solution for $\dot{\psi}(t)$ to get $\psi(t) = \int \dot{\psi}(t)dt$. Here the integration is with respect to time. Fourth, combine steps one and three to get an explicit solution $F(y, t) = c$ for some arbitrary constant c .

Let us next consider cases of higher-order differential equations. First consider the case of second-order linear, homogeneous differential equations with constant coefficients. Such equations are of the form:

$$\ddot{y} + a_1\dot{y} + a_0y = 0$$

Let us guess a solution of the form, $y = ce^{\lambda t}$. Earlier study of the case of the first-order linear, homogeneous differential equation with constant coefficients. Therefore, if this solution is correct, we also know that $\dot{y} = c\lambda e^{\lambda t}$ and $\ddot{y} = c\lambda^2 e^{\lambda t}$. Plugging into the original differential equation, we find that:

$$\begin{aligned}\ddot{y} + a\dot{y} + by &= c\lambda^2 e^{\lambda t} + a_1 c \lambda e^{\lambda t} + a_0 c e^{\lambda t} \\ &= (\lambda^2 + a_1 \lambda + a_0) c e^{\lambda t} = 0,\end{aligned}$$

if and only if $(\lambda^2 + a_1 \lambda + a_0) = 0$, for $c \neq 0, e^{\lambda t} \neq 0$.

The polynomial expression, $\lambda^2 + a_1 \lambda + a_0 = 0$, has two solutions, λ_1 and λ_2 . The general solution is then given by:

$$y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

The values of c_1 and c_2 can be obtained from initial conditions. Solutions of such form are straightforward, unless there are repeated root or complex roots to the quadratic form, $\lambda^2 + a_1 \lambda + a_0 = 0$. We will deal with cases a little later.

Finally, consider the case of higher-order differential equations. The simplest case is linear homogenous, with constant coefficients. Consider the equation:

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + a_{n-2} \frac{d^{n-2} y}{dt^{n-2}} + \dots + a_1 \frac{dy}{dt} + a_0 y(t) = 0$$

Some of the coefficients may be zero, but not the coefficient on $\frac{d^n y}{dt^n}$, or else it would be an equation of the $(n-1)^{st}$ order. Given that the coefficient on $\frac{d^n y}{dt^n}$ is non-zero, it has been normalized to one and all other coefficients adjusted accordingly. Note that for such an equation, you will need n initial conditions to define it precisely. Typically higher-order derivatives will be used to pin down the exact form of y . The solution is of the form:

$$y(t) = \sum_{i=1}^n c_i e^{\lambda_i t}$$

where the λ_i are the n roots to the polynomial

$$\lambda^n + \sum_{i=1}^{n-1} a_{n-i} \lambda^{n-i} + a_0 = 0$$

There are three separate cases. Suppose first that each of the n roots is real and distinct. Then the solution is $y(t) = \sum_{i=1}^n c_i e^{\lambda_i t}$, where the c_i are can be determined using

the n initial conditions. Suppose second that there are repeated real roots. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be real roots having respective multiplicities m_1, m_2, \dots, m_k . Then the solution is $y(t) = \sum_{i=0}^{m_1-1} c_{i,1} t^i e^{\lambda_1 t} + \sum_{i=0}^{m_2-1} c_{i,2} t^i e^{\lambda_2 t} + \dots + \sum_{i=0}^{m_k-1} c_{i,k} t^i e^{\lambda_k t}$, and the constants are determined using the initial conditions.

Suppose now that some of the roots are complex. First, ignore the complex roots and use the arguments above to deal with the real roots. Now let's think about the

complex roots, which always occur in conjugate pairs. So to have a repeated complex root, one would have to have a fourth order or higher differential equation, and these rarely occur in economics.

So suppose that you have a single pair of complex roots. Let these be $a + bi$ and $a - bi$. Euler's formula states that:

$$e^{i\theta} = \cos\theta + i\sin\theta, \text{ where } i \equiv \sqrt{-1}$$

We can begin naively by writing down:

$$c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} = c_1 e^{at}(\cos bt + i\sin bt) + c_2 e^{at}(\cos(-bt) + i\sin(-bt)).$$

Since $\cos(-bt) = \cos bt$ and $\sin(-bt) = -\sin bt$, we see that:

$$c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} = (c_1 + c_2)e^{at} \cos bt + (c_1 - c_2)ie^{at} \sin bt.$$

If we are thinking in the complex plane, i is just a particular constant. Rewriting the above equation using these new constants: $c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t} = e^{at}[k_1 \cos bt + k_2 \sin bt]$. Now it is easy to generalize. Suppose the pair of roots $a + bi$ and $a - bi$ appears exactly m times. Then, again for the scalar case:

$$y_I = e^{at} \left[\sum_{i=0}^{m-1} k_{1,i} t^i \cos bt + \sum_{i=0}^{m-1} k_{2,i} t^i \sin bt \right]$$

The arguments for the system of equations also go through analogously.

These principles generalize for non-homogenous linear equations with constant coefficients. Thus suppose the equation is $y^{(n)} + \sum_{i=1}^n a_{n-i} y^{(n-i)} = g(t)$. First, solve for

the complementary function using the solution to the homogeneous system. The solution to the particular integral is not so easy now. An important generic technique is called the method of undetermined coefficients.

If $g(t)$ is a polynomial, an exponential function, or a trigonometric function, or a linear combination of these, we can find a particular integral. Let us assume it is only one of these; the technique will easily generalize. For example, if the forcing term is a

polynomial $g(t) = \sum_{i=0}^k b_i t^i = P_k(t)$, then $y_p = t^s \sum_{i=0}^k b_i t^i$, where s is the multiplicity of 0 as

a root in the characteristic equation. Then use the original definition of the differential equation to solve for the coefficients b_0, \dots, b_k . Assume now that $g(t) = P_k(t) \exp(\alpha t)$, and let s be the multiplicity of α as a root in the characteristic equation. Then the conjectured particular solution $y_p = t^s P_k(t) \exp(\alpha t)$, and again you can use the original differential equation to solve for the b_0, \dots, b_k . Now if $g(t) = P_k(t) e^{\alpha t} \sin \beta t$ or $P_k(t) e^{\alpha t} \cos \beta t$. and s is the multiplicity of $\alpha + \beta i$ as a root in the characteristic equation,

then the conjectured particular solution $y_p = t^s \sum_{i=0}^k b_i t^i e^{at} \cos \beta t + t^s \sum_{i=0}^k b_i t^i e^{at} \sin \beta t$. The adjustment by the factor s ensures that the particular solution is linearly independent of the homogenous solution.

The variation of parameters procedure also generalizes to higher-order differential equations. This procedure can get quite messy. However, it remains relatively tractable at least for second-order equations. For the differential equation, $\ddot{y} + a_1 \dot{y} + a_0 y = 0$, we have the complementary function:

$$y_c(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \equiv y_1(t) + y_2(t).$$

We again assume that $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$. Now plugging $u_1(t)y_1(t) + u_2(t)y_2(t)$, and its first and second derivatives into the original equation and simplifying, we find that :

$$\dot{u}_1 \dot{y}_1 + \dot{u}_2 \dot{y}_2 = b(t)$$

We still have two unknown functions, u_1 and u_2 , and only one equation. It turns out that we can add a second, *arbitrary* equation. A convenient second equation is $\dot{u}_1 y_1 + \dot{u}_2 y_2 = 0$. These simultaneous equations may be solved as:

$$\dot{u}_1 = -\frac{b(t)y_2(t)}{(y_1 \dot{y}_2 - \dot{y}_1 y_2)}$$

$$\dot{u}_2 = \frac{b(t)y_1(t)}{(y_1 \dot{y}_2 - \dot{y}_1 y_2)}$$

The right-hand sides of these equations are functions only of time, and therefore they may be integrated to solve for $u_1(t)$ and $u_2(t)$.

Even if one conjectures a clever particular solution, it is often difficult to tell if it is independent of the homogenous solution, especially if $g(t)$ does not fall explicitly into one of the three above categories. Accordingly, it is always acceptable to form the Wronskian and calculate its determinant. For more details on how to use the Wronskian, see Boyce, W. E. and R. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (8th ed.), New York: Wiley, 2005, Chapter 3.

Finally, the general solution is just $y = y_c + y_p$.