The American Problem

Maoyong is now the head of an American company in China. His new boss asks, ”How would your answer change if the Prime Minister told you that you had a 50-year monopoly, but you could leave some oil in the ground if you want?” Is it possible that, with a given time frame like the 50 year plan, it might be optimal not to sell all of the oil? A monopolist typically finds it optimal to restrict output to maximize profits. Maybe it is better to leave some of the oil in the ground at the end of the fifty years to maintain a higher price.

Maoyong recognizes that the solution to the problem must again satisfy the first-order conditions:

$$\frac{\partial H}{\partial u} = 0, \ \frac{\partial H}{\partial y} = -\dot{\lambda}, \ \dot{y} = f(t, y, u)$$

Maoyong is still constrained by $y(0) = 100$. Maoyong must also end up with a non-negative amount of oil in the ground at the end of his 50 year planning horizon. He replaces his original terminal value condition with the complementary slackness condition:

$$\lambda(T) \geq 0, \ (y(T) - y_{\min}(T)) \ \geq 0, \ \lambda(T)(y(T) - y_{\min}(T)) \ = 0$$

Maoyong first realizes that the original first-order conditions must hold, including the result of a time-constant costate variable, λ^*. Therefore:

$$y^*(t) = \hat{c} + \frac{\lambda^*}{2\rho} e^{\rho t} - \frac{t}{2}$$

Now the solution must also satisfy the complementary slackness condition. There are only two possibilities. First, we may have $\lambda^* = 0$ and $y(T) > 0$, or in this specific case, $y(50) > 0$. Let us therefore provisionally assume that $\lambda^* = 0$. We obtain:

$$y^*(t) = \hat{c} - \frac{t}{2}$$

We still require that $y(0) = 100$, and so:

$$y^*(0) = 100 = \hat{c}$$

Thereore:

$$y^*(t) = 100 - \frac{t}{2},$$

and $u^*(t) = -\dot{y}^*(t) = \frac{1}{2}$

For this specific case, we now find that $y(50) = 100 - 25 = 75 > 0$. Our guess that $y(50) > 0$, turns out to be true. However, in general, we might find that the required value of $y(T)$ for $\lambda^* = 0$ may turn out to be negative. As long as $T < 200$, $y^*(T) > 0$. Alternatively, suppose that $T > 200$. In this case $\lambda^* = 0$ would require that $y(T) < 0$. Therefore, our presumption that the $y(T) > 0$ cannot be correct. In this case it must be true that $y(T) = 0$ and $\lambda(T) = \lambda^* > 0$. As before we now use the terminal condition, $y(T) = 0$ to pin down the constants in the solution equation. Therefore:

$$y_0 = 100 = \hat{c} + \frac{\lambda^*}{2\rho}$$

and

$$y_T = 0 = \hat{c} + \frac{\lambda^*}{2\rho}\exp(\rho T) - \frac{T}{2}$$

Solving these two simultaneous equations, we find that:

$$\hat{c} = -\frac{[100e^{\rho T} - \frac{T}{2}]}{1 - e^{\rho T}}$$

$$\lambda^* = \frac{2\rho[100 - \frac{T}{2}]}{1 - e^{\rho T}}$$

Note that for $T > 200$, $\lambda^* > 0$. This last result confirms that Maoyong would actually prefer to ignore the constraint by setting $y(T) < 0$.

For Maoyong, when $T = 50$, $y^*(50) = 75$. Therefore Maoyong's American bosses are content to leave most of the oil in the ground to avoid flooding the market with too much oil. Recall that Maoyong's revenue at each point in time, is given by:

$$u(t)[1 - u(t)]$$

Marginal revenue equals $1 - 2u(t)$. As there are no extraction cost, marginal cost is zero.
Maoyong thinks to himself, "This solution makes sense because now I can be a monopolist for as long as they let me, and every Buckeye knows that a monopolist produces $u(t) = 1/2$ since that is the point where marginal revenue equals marginal cost."

To see how non-negativity constraints may come into play, let us return to Maoyong's American Problem. However, let us now assume that his American boss gives him a 250 year time horizon. Of course, neither Maoyong or his American boss are likely to be able to do an after-the-fact review of the results, but potential stockholders realize that the price of a stock should reflect the present value of profits over an infinite horizon. If we ignore the constraint in this example, the solution to the problem requires the selection of $y(250) = -25$. Maoyong now realizes that the non-negativity constraint must be strictly binding. You can quickly verify that:

$$y^*(t) = \begin{cases} \frac{100e^{\rho 250} - 125 + 25e^{\rho t}}{e^{\rho 250} - 1} - \frac{t}{2},\ 0 \le t \le 250 \\ 0,\ t > 250 \end{cases}$$