

Bailey's Measure of the Welfare Costs of Inflation as a General-Equilibrium Measure*

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Abstract

Lucas (2000) has shown that Bailey's formula for the welfare costs of inflation can be regarded as an approximation to the general-equilibrium measures which emerge from the Sidrauski and the shopping-time models. In this paper we show that Bailey's measure can be exactly obtained in the Sidrauski general-equilibrium framework under the assumption of quasilinear preferences. The result, based on whether or not wealth effects are incorporated into the analysis, is also helpful in clarifying why Lucas' measure derived from the Sidrauski model turns out to be an upper bound to Bailey's. Two examples are used to illustrate the main conclusions.

1 Introduction

Since the seminal work of Bailey (1956), economists have devoted considerable effort to measuring the welfare cost of inflation. Bailey's formula has been derived in a partial-equilibrium setting. It treats real money balances as a consumption good and inflation as a tax on real balances. The welfare cost is computed by measuring the area under the inverse money demand function.

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Lucas (2000) has shown that, for small values of the nominal interest rate, Bailey's measure happens to give a good approximation to the general-equilibrium measures that emerge from the Sidrauski and the shopping-time models.

The purpose of this paper is to show that Bailey's formula can actually be obtained in Sidrauski's general-equilibrium framework as the correct measure of the welfare costs of inflation by considering a specific class of utility functions. As stated by Gillman (1995, p.1): "evidence suggests that integration under the money demand function appears applicable in general-equilibrium economies". Based on a specific utility, this conjecture is proved here.

This point is important because Bailey's methodology has served as the basis for several well-known measurements of the welfare costs of inflation in the economic literature (e.g., Bailey (1956), Lucas (1981) and Fisher (1981), among many others).

Moreover, until this day one can still find many measurements of such welfare costs based on Bailey's formula (e.g., Lucas (2000), Attanasio et al. (2002), Serletis and Virk (2006), Craig and Rocheteau (2006) and Ireland (2007)), and therefore, all such measurements are subject to the usual criticism that they originate from a partial-equilibrium analysis¹.

This paper provides a counterexample to this usual proviso that Bailey's measure is to be regarded as no more than a partial-equilibrium measure. For a specific class of utility functions, we show that Bailey's measure provides an exact measure of the welfare costs of inflation derived from an intertemporal general-equilibrium model of the entire economy, rather than just from a partial-equilibrium analysis of one single market.

2 The Model

To simplify the exposition, we present Sidrauski's model in a continuous-time setting and abstract from growth considerations. Real output is normalized to 1.

We shall write M for money, B for bonds, P for the price index, and c , m , and h , respectively, for real consumption, real money balances and real lump-sum taxes. The amount rB stands for the nominal interests paid by bonds to the representative consumer.

¹For instance, one reads in footnote 10 of Craig and Rocheteau: "The Bailey analysis is subject to the following caveats. It is a partial equilibrium analysis that assumes away externalities, *general equilibrium effects*, and distributional effects (the italics have been placed by this author) ."

Given $g > 0$, the representative consumer is supposed to derive utility from the maximization of real consumption and real balances according to:

$$\int_0^{\infty} e^{-gt} U(c, m) dt \quad , \quad g > 0 \quad (1)$$

where U is a concave utility function, strictly increasing in both c and m . The maximization is subject to the budget constraint:

$$\dot{M} + \dot{B} = P - Pc - Ph + rB \quad (2)$$

The dot over the variable stands for its derivative with respect to time. Divide both members of (2) by P and use the formula of the derivative of a fraction to obtain the budget constraint as a function of $b = B/P$, c , m and h :

$$\dot{b} + \dot{m} = 1 - c - h - \pi m + (r - \pi)b \quad (3)$$

In (3), π denotes the rate of inflation.

Solve for c in (3), substitute it in (1) and use calculus of variations (Euler equations with respect to b, \dot{b}, m, \dot{m}) to obtain the following usual steady-state first-order conditions:

$$r = \pi + g \quad (4)$$

$$\frac{U_c}{1} = \frac{U_m}{r} \quad (5)$$

Given the assumed concavity of the utility function, such conditions are also sufficient for optimality.

Equilibrium in the goods market implies:

$$c(t) = 1, \forall t \quad (6)$$

In the steady-state $\dot{b} = \dot{m} = 0$ and the government budget equation reads:

$$h + \pi m = (r - \pi)b \quad (7)$$

In the steady-state inflation equals the rate of monetary expansion (μ).

As in Lucas (2000), we define the welfare cost of a positive nominal rate r (by now, call it $w(r)$) as the additional income the representative agent has to receive in order to be indifferent between an interest rate equal to r and an interest rate equal to 0:

$$U(1 + w(r), m(r)) = U(1, m(0)) \quad (8)$$

Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a strictly increasing and concave function. Lucas assumes a utility function belonging to a class \mathbb{U}_1 defined by:

$$U \in \mathbb{U}_1 \iff U(c, m) = \frac{1}{1 - \sigma} \left[c \varphi\left(\frac{m}{c}\right) \right]^{1 - \sigma} \quad (9)$$

$\sigma > 0, \sigma \neq 1$. In this case, using (5) leads precisely to equation (3.7) of Lucas' paper²:

$$\psi\left(\frac{m}{c}\right) = \frac{\varphi'\left(\frac{m}{c}\right)}{\varphi\left(\frac{m}{c}\right) - \frac{m}{c}\varphi'\left(\frac{m}{c}\right)}, \quad (10)$$

in which $\psi(\cdot)$ stands for the inverse of the money demand function $m(r)$. It can be easily shown that:

$$\psi'\left(\frac{m}{c}\right) = \frac{\varphi\left(\frac{m}{c}\right)\varphi''\left(\frac{m}{c}\right)}{\left(\varphi\left(\frac{m}{c}\right) - \frac{m}{c}\varphi'\left(\frac{m}{c}\right)\right)^2} < 0, \quad (11)$$

a fact we use below.

Let $w_1(r)$ stand for the measure of the welfare costs of inflation when the utility function belongs to the class defined by \mathbb{U}_1 . Lucas shows that $w_1(r)$ is given by his differential equation (equation (3.11) in Lucas (2000)):

$$w_1'(r) = -\psi\left(\frac{m(r)}{1 + w_1(r)}\right)m'(r), \quad w_1(0) = 0 \quad (12)$$

Now, let $B(r)$ denote Bailey's partial-equilibrium measure of the welfare costs of inflation and consider a given money demand function $m(r)$:

$$B'(r) = -rm'(r), \quad B(0) = 0 \quad (13)$$

Note that, since $\frac{m(r)}{1 + w_1(r)} < m(r)$, and $\psi'(m) < 0$:

$$\psi\left(\frac{m(r)}{1 + w_1(r)}\right) > r \quad (14)$$

Since $m'(r) < 0$, (12) and (14) imply:

$$B(r) < w_1(r) \quad (15)$$

²Lucas writes equations (10) and (11) below using (6). We continue using c in the denominator because it may be helpful to the reader to understand the intuition behind our results. See below.

for all strictly positive r . In other words, Lucas' general-equilibrium measure of the welfare costs of inflation that emerges from Sidrauski's model and a utility function belonging to the class defined by (9) turns out to be an upper bound to Bailey's measure³.

Example 1 *Just to illustrate inequality (15), let us calculate Lucas' measure (12), assuming an utility function belonging to the class defined by (9), in the usual case of the log-log money demand $m = Kr^{-\alpha}$, $\alpha > 0$ (which has been used, for instance, by Lucas (2000, sections 3 and 5), with $\alpha = 0.5$). Since in this case $\psi(m) = \left(\frac{K}{m}\right)^{1/\alpha}$, (12) leads to:*

$$\int_0^{w_1(r)} (1 + \nu)^{-1/\alpha} d\nu = \int_0^r \alpha K \rho^{-\alpha} d\rho$$

By integration:

$$w_1(r) = -1 + (1 - Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}}. \quad (16)$$

Using Bailey's formula we get:

$$B(r) = \frac{\alpha K}{1 - \alpha} r^{1-\alpha} \quad (17)$$

We know from Bernoulli's generalized inequality⁴ that for any: $x \in (-1, 0) \cup (0, +\infty)$ and any real number $\gamma \in (-\infty, 0) \cup (1, +\infty)$:

$$(1 + x)^\gamma > 1 + x\gamma$$

In order for $w_1(r)$ to exist and be positive we must have $r \in \left(0, K^{\frac{1}{\alpha-1}}\right)$. Since $-Kr^{1-\alpha} \in (-1, 0)$ and $\frac{\alpha}{\alpha-1} \in (-\infty, 0) \cup (1, +\infty)$, Bernoulli's inequality applies, yielding:

$$(1 + (-Kr^{1-\alpha}))^{\frac{\alpha}{\alpha-1}} > 1 + (-Kr^{1-\alpha}) \frac{\alpha}{\alpha-1}$$

which is equivalent to

$$w_1(r) = -1 + (1 - Kr^{1-\alpha})^{\frac{\alpha}{\alpha-1}} > \frac{\alpha K}{1 - \alpha} r^{1-\alpha} = B(r)$$

This example illustrates, in this particular case, the general result given by (15).

³This result is extended to the case of an economy with many types of monies in Cysne and Turchick (2007). See also Proposition 2 in Cysne (2003) for a generalized comparison between Bailey's measure and Lucas' measure of the welfare costs of inflation when the latter is derived from a shopping-time model.

⁴See, e.g., Mitrovic (1970).

It is important to understand the intuition behind (15). The upshot is that the measure w_1 , based on $\psi(\frac{m(r)}{1+w_1(r)})$, rather than on $r = \psi(\frac{m(r)}{1})$, takes into consideration the general-equilibrium effect arising out of the fact that the consumer, in the new steady state, is given an extra income $w_1(r)$. Bailey's formula does not incorporate this effect.

Note from (11) that, for a given money-demand function, the equilibrium interest rate is a positive function of consumption. This makes $r = \psi(\frac{m(r)}{1}) < \psi(\frac{m(r)}{1+w_1(r)})$, the fact which, as we have seen above, leads to $B(r) < w_1(r)$.

The remark above provides a hint in the search for an alternative class of utility function (call it \mathbb{U}_2) under which Bailey's formula can emerge as an exact general-equilibrium measure of the welfare costs of inflation. This should be a class leading to money-demand functions insensitive to income changes. One should have $\frac{\partial(\frac{U_m}{U_c})}{\partial c} = 0$, which leads to:

$$U_m U_{cc} = U_c U_{cm} \tag{18}$$

3 Bailey's Welfare Measure as a General Equilibrium Measure

Consider now any utility function belonging to the alternative class \mathbb{U}_2 of concave monotonic transformations of the quasi-linear utility function $c + \lambda(m)$:

$$U \in \mathbb{U}_2 \iff U(c, m) = g(c + \lambda(m)) \tag{19}$$

with $g' > 0$, $g'' \leq 0$, $\lambda'(m) > 0$ and $\lambda''(m) < 0$ ⁵. It is also permitted for λ to be strictly increasing and strictly concave only over an interval $(0, K)$, and not from point K on.

Note that if $U \in \mathbb{U}_2$, it trivially satisfies (18).

Any U in this class of functions is concave in (c, m) , since it is given by the composition of two concave and increasing functions. This makes $e^{-gt}U$ concave with respect to (b, \dot{b}, m, \dot{m}) , in which case the Euler equations are sufficient for an optimum⁶.

⁵An example of utilization of such quasi-linear preferences in macroeconomics is given by Greenwood et al. (1988). The purpose of the authors in using such a specification, in that case, is abstracting from wealth effects in labor supply.

⁶We assume the parameters of the problem to be such that interior solutions are generated.

Let $w_2(r)$ denote the measure of the welfare cost of inflation which emerges from the Sidrauski model when the utility function belongs to the class defined by \mathbb{U}_2 .

Proposition 1 below presents our main result. It shows that, for all $U \in \mathbb{U}_2$, Bailey's formula provides an exact general-equilibrium measure of the welfare costs of inflation.

Proposition 1 *Let an economy be described as above, with utility function $U \in \mathbb{U}_2$. Then,*

$$w_2(r) = B(r)$$

Proof. Use (5) and (19) to obtain:

$$\psi(m) := r = \lambda'(m) \tag{20}$$

This equation (which stands for the household's preferences) determines an inverse money demand function $\psi(m)$ with $\psi'(m) = \lambda''(m) < 0$ which, in contrast to the one obtained by Lucas, does not depend on the value that c assumes in the steady state (if $c = 1$ or $c = 1 + w_2(r)$).

Plugging (19) into (8)⁷:

$$w_2(r) + \lambda(m(r)) = \lambda(m(0)) \tag{21}$$

Differentiating both sides with respect to the interest rate r :

$$w_2'(r) + \lambda'(m)m'(r) = 0 \tag{22}$$

From (20) and (22):

$$w_2'(r) = -rm'(r) = B'(r) \tag{23}$$

Since $w_2(0) = B(0) = 0$, $w_2(r) = B(r)$ for any value of r . ■

Example 2 *Take $U \in \mathbb{U}_2$ such that, when U is written like in (19), $\lambda(m) = \frac{m}{\alpha} \left(1 + \log\left(\frac{K}{m}\right)\right)$, where $\alpha, K > 0$. This λ is strictly increasing and concave on the interval $(0, K)$.*

Equation (20) gives

$$r = \lambda'(m) = \frac{1}{\alpha} \log\left(\frac{K}{m}\right),$$

which corresponds to the semi-log money-demand function,

$$m = Ke^{-\alpha r}.$$

⁷ $m(0)$ should read K in case $(0, K)$ is the maximal interval over which λ is strictly increasing and concave.

It is instructive to compare the usual calculation of Bailey's welfare measure for this case,

$$\begin{aligned} B(r) &= - \int_0^r \rho m'(\rho) d\rho = K \int_0^r \rho \alpha e^{-\alpha\rho} d\rho = K \left[-\rho e^{-\alpha\rho} \Big|_0^r + \int_0^r e^{-\alpha\rho} d\rho \right] \\ &= K \left[-r e^{-\alpha r} - \frac{e^{-\alpha r}}{\alpha} + \frac{1}{\alpha} \right], \end{aligned}$$

Alternatively, using the formula provided by (21):

$$w_2(r) = \lambda(K) - \lambda(Ke^{-\alpha r}) = \frac{K}{\alpha} - \frac{Ke^{-\alpha r}}{\alpha} (1 + \alpha r)$$

From the two last equations above we have $w_2(r) = B(r)$, illustrating Proposition 1. In this case, actually, (21) and Proposition 1 turn out to provide an easier way of calculating B .

Note that (15) does not bear any direct connection to the well known inequality from partial-equilibrium microeconomics where, for normal goods, the equivalent variation is no less than the consumers' surplus (see, e.g., Mas-Colell et al. (1995), p. 82). Indeed, this inequality ranks welfare variations, whereas (15) compares deadweight losses.

The result established by Proposition 1, however, does share with standard partial-equilibrium microeconomics the fact that, when utility is quasilinear (in which case $g(\cdot)$ in (19) is equal to the identity function), thereby ruling out wealth effects, the area under the Marshallian demand function (here, Bailey's measure) gives an exact measure of the deadweight loss arising from taxation⁸.

4 Conclusions

Lucas (2000) has shown that Bailey's formula for the welfare costs of inflation can be regarded as a very good approximation to general-equilibrium measures originating from the shopping-time and the Sidrauski models. Deepening such a result in this paper we show that, under quasilinear preferences,

⁸Note that in our intertemporal context a concave monotonic transformation of a quasilinear function (like in (19)) is sufficient for wealth effects to be ruled out. Preferences are quasilinear (see Definition 3.B.7 in Mas-Colell et al.), but not necessarily the utility function.

Bailey's formula provides an exact measure of the welfare costs of inflation in a Sidrauski general-equilibrium framework.

In this particular case, therefore, cautioning the reader about measurements of the welfare costs of inflation based on Bailey's formulas as being "the outcome of just a partial-equilibrium analysis", as one sometimes finds in the literature, is not necessary. Although variations in relative prices and income effects are ruled out in the simplified type of general-equilibrium model worked out here, we have shown that Bailey's formula can emerge from a model which considers the entire economy (including the process of money creation and the endogenous determination of the money demand and of the nominal interest rate), rather than just one single market.

The intuition underlying such a result is the usual one from standard microeconomics: in the absence of wealth effects, the Marshallian area-under-the-inverse-demand-curve measure, rather than an approximation, turns out to provide an exact measure of the deadweight loss stemming from taxation.

The result also clarifies why the Sidrauski-Lucas' measure of the welfare costs of inflation turns out to be an upper bound to Bailey's.

Two examples based were offered to illustrate the results.

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