Sums of exponentials of random walks with drift

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Abstract

For many time series in empirical macro and finance, it is assumed that the logarithm of the series is a unit root process. Since we may want to assume a stable growth rate for the macroeconomics time series, it seems natural to potentially model such a series as a unit root process with drift. This assumption implies that the level of such a time series is the exponential of a unit root process with drift and therefore, it is of substantial interest to investigate analytically the behavior of the exponential of a unit root process with drift. This paper shows that the sum of the exponential of a random walk with drift converges in distribution, after rescaling by the exponential of the maximum value of the random walk process. A similar result was established in earlier work for a unit root processes without drift. The results derived here suggest the conjecture that also in the case when the Dickey-Fuller test or the KPSS statistic is applied to the exponential of a unit root process with drift, these tests will asymptotically indicate stationarity.

Key words and phrases: nonlinear transformation of random walk, unit roots

1 Introduction

This paper concerns the behavior of the exponential of a random walk process with drift. Since the logarithms of time series such as GDP, money supply, consumption, unemployment rate, and interest rate have been modeled in the literature as unit root processes, it is of

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considerable interest to investigate the level behavior of such time series. This paper shows for the random walk with drift case the convergence in distribution of statistics of the form

$$R_n = \sum_{t=1}^{n} \exp(y_t - M_n)$$

where y_t, $t = 1, \ldots, n$ is a random walk, $y_0 = 0$, and $M_n = \max_{1 \le t \le n} y_t$.

When y_t is a unit root process without drift, the result of convergence in distribution of R_n was shown in de Jong (2010). He considered a more general situation than the case of i.i.d. increments Δy_t considered in this paper, and assumed that the increments of the unit root process were linear processes. The assumption of i.i.d. Δy_t is much stronger than the linear process assumption made in de Jong (2010). However, we do not know how to relax this i.i.d. assumption for the case of a unit root process with drift as considered in this paper. On the other hand, we do not need the assumption of continuity of the innovation distribution that is made in de Jong (2010).

For the case of i.i.d increments Δy_t and no drift (i.e., $E\Delta y_t = 0$), Davies and Krämer (2003) showed the property $\sup_{n \ge 1} ER_n < \infty$ under regularity conditions. For this case Park and Phillips (1999) already showed that under regularity conditions,

$$n^{-1/2} \sum_{t=1}^{n} \exp(y_t - M_n) \xrightarrow{p} 0;$$

see also Davies and Krämer (2003, p.867)[1].

The key argument in this paper consists of showing that

$$\lim_{n \to \infty} E \exp(-r \sum_{t=1}^{n} \exp(y_t - M_n))$$

exists for all $r \ge 0$. In combination with the result $\sum_{t=1}^{n} \exp(y_t - M_n) = O_p(1)$, this then completes the proof of convergence in distribution. The proof exploits the assumption of i.i.d increments to split $\sum_{t=1}^{n} \exp(y_t - M_n)$ into two parts that possess a property of mutual independence (one running up to the index at which the maximum of the random walk is attained, and one starting at that point), and then shows the convergence of each part to obtain the convergence in distribution of R_n. De Jong (2010) used a different approach, based on a result of Ritter (1981). Ritter (1981) establishes a rate at which a "random walk conditioned to be positive" increases to infinity. In the absence of a similar result for random walk with drift, this strategy of proof is not available for the case of random walk with drift.

[1]In their paper, $M_n = \max(0, \max_{1 \le t \le n} y_t)$.

Establishing a Ritter-type result for random walk with drift appears to be complex, and we were unable to derive such a result.

Earlier attempts to analyze issues involving transformations of unit root processes are Granger and Hallmann (1991), Ermini and Granger (1993), Corradi (1995), and Ermini and Hendry (2008). A common strategy of these papers is to try to define the I(1) property in such a way that under some transformations the property is preserved. A related literature (see for example Granger and Hallmann (1991), Burridge and Guerre (1996), and Gourieroux and Breitung (1997)) seeks to find unit root tests whose null distribution is robust to monotonic transformations.

Borodin and Ibragimov (1995) and Park and Phillips (1999) seek to characterize the limit behavior (after rescaling) of sums of the form

$$\sum_{t=1}^{n} f(y_t)$$

for various types of functions $f(\cdot)$. For functions $f(\cdot)$ that are "asymptotically homogeneous" the limit behavior of the rescaled statistic can be derived because it is asymptotically equivalent to a sum of a function of $y_t/\sqrt{T}$, and at that point an appeal to the functional central limit theorem can be used to derive the limit distribution. For integrable functions $f(\cdot)$, the convergence in distribution of $n^{-1/2}\sum_{t=1}^{n} f(y_t)$ has been derived in Borodin and Ibragimov (1995) and Park and Phillips (1999). De Jong (2010) adds the exponential function to the set of functions for which the behavior of statistics such as $\sum_{t=1}^{n} f(y_t)$ is well understood.

Since a random walk with positive drift is not recurrent, the results of this paper immediately carry over to the statistic

$$\sum_{t=1}^{n} \exp(y_0 + y_t - \max(0, y_0 + \max_{1\leq t\leq n} y_t)),$$

where y_0 is an arbitrary random variable. In addition, the line of proof used here carries over to statistics of the form

$$\sum_{t=1}^{n} f(y_t - \max_{1\leq t\leq n} y_t)$$

and

$$\sum_{t=1}^{n} f(y_0 + y_t - \max(0, y_0 + \max_{1\leq t\leq n} y_t)),$$

where $f : (-\infty, \infty) \to [0, \infty)$ is a Borel measurable function such that $f(x)$ is continuous and nondecreasing in x, and $|f(x)| \leq C|x + 1|^{-2-\eta}$ for all $x \leq 0$ and some $\eta > 0$.

2 Main result

The random walk with drift y_t is assumed to satisfy the following:

Assumption 1. *$y_t = y_{t-1} + \alpha + \varepsilon_t$ for $t = 1, ..., n$, where $\alpha > 0$, $y_0 = 0$, ε_t is i.i.d., $E\varepsilon_t = 0$ and $E|\varepsilon_t|^{2+\delta} < \infty$ for some $\delta > 0$.*

To understand the development of the result, define T_n as the first index at which the maximum of y_t is attained. We can write, if we adopt the conventions that summations over empty index sets equal 0, that a minimum over an empty set equals $+\infty$, and that a maximum over an empty set equals $-\infty$,

$$
\begin{aligned}
&E\exp(-r\sum_{t=1}^{n}\exp(y_t - M_n)) \\
&= E\sum_{k=1}^{n} I(T_n = k)\exp(-r\sum_{t=1}^{n}\exp(y_t - y_k)) \\
&= \sum_{k=1}^{n} E(I(\min_{1\le s\le k-1}(y_k - y_s) > 0)I(\min_{k+1\le s\le n}(y_k - y_s) \ge 0)\exp(-r\sum_{t=1}^{n}\exp(y_t - y_k))) \\
&= \exp(-r)\sum_{k=1}^{n} E(\exp(-r\sum_{t=1}^{k-1}\exp(y_t - y_k))I(\min_{1\le s\le k-1}(y_k - y_s) > 0)) \\
&\qquad \times E(\exp(-r\sum_{t=k+1}^{n}\exp(y_t - y_k))I(\min_{k+1\le s\le n}(y_k - y_s) \ge 0)) \\
&= \exp(-r)\sum_{k=1}^{n} E(\exp(-r\sum_{t=1}^{k-1}\exp(-y_t))I(\min_{1\le s\le k-1} y_s > 0)) \\
&\qquad \times E(\exp(-r\sum_{t=1}^{n-k}\exp(y_t))I(\max_{1\le s\le n-k} y_s \le 0)) \\
&= \exp(-r)\sum_{k=1}^{n} u_{k-1}v_{n-k}
\end{aligned}
$$

where $u_0 = v_0 = 1$, and for $1 \le k \le n$,

$$
u_k = E(\exp(-r\sum_{t=1}^{k}\exp(-y_t))I(\min_{1\le s\le k} y_s > 0))
$$

and

$$v_k = E(\exp(-r\sum_{t=1}^{k}\exp(y_t))I(\max_{1\leq s\leq k} y_s \leq 0)).$$

The third equality follows because the two parts are independent. The fourth equality follows because for any Borel measurable function $h(.,\dots,.)$,

$$\begin{aligned}&h(\alpha+\varepsilon_k, 2\alpha+\varepsilon_k+\varepsilon_{k-1},\cdots,(k-1)\alpha+\varepsilon_k+\varepsilon_{k-1}+\cdots\varepsilon_2)\\ &\stackrel{d}{=} h(\alpha+\varepsilon_1, 2\alpha+\varepsilon_1+\varepsilon_2,\cdots,(k-1)\alpha+\varepsilon_1+\varepsilon_2+\cdots\varepsilon_{k-1})\end{aligned}$$

and

$$\begin{aligned}&h(\alpha+\varepsilon_{k+1}, 2\alpha+\varepsilon_{k+1}+\varepsilon_{k+2},\cdots,(n-k)\alpha+\varepsilon_{k+1}+\varepsilon_{k+2}+\cdots\varepsilon_n)\\ &\stackrel{d}{=} h(\alpha+\varepsilon_1, 2\alpha+\varepsilon_1+\varepsilon_2,\cdots,(n-k)\alpha+\varepsilon_1+\varepsilon_2+\cdots\varepsilon_{n-k})\end{aligned}$$

Now we can show the following results:

Lemma 1. *Under Assumption 1,* $\sup_{n\geq 1} ER_n < \infty$.

Lemma 2. *Under Assumption 1,* u_n *and* $\sum_{k=1}^{n} v_{n-k}$ *converge as* $n \to \infty$.

Using Lemma 1 and 2, the following result now follows:

Theorem 1. *Under Assumption 1,* R_n *converges in distribution.*

For completeness, we note that the case of the unit root process with negative drift - i.e. the case $\alpha < 0$ - is substantially simpler to analyze. In that case, by the root test, $\sum_{t=1}^{n}\exp(y_t)$ will converge almost surely if $\lim_{n\to\infty}\exp(y_n/n) < 1$ a.s.. Therefore, as long as the strong law of large numbers holds for $n^{-1}\sum_{t=1}^{n}\varepsilon_t$, we will find $\lim_{n\to\infty}\exp(y_n/n) = \exp(\alpha) < 1$ a.s., and therefore almost sure convergence of $\sum_{t=1}^{n}\exp(y_t)$ follows.

While Theorem 1 shows the convergence in distribution, the limit distribution in general depends on the distribution on ε_t. Therefore, if the above limit distribution result is to be used for testing, the limit distribution will need to be obtained through some resampling method.

We conjecture that the result of Theorem 1 will continue to hold for linear processes. It should be possible to obtain such a result by following the line of proof of de Jong (2010), in combination with a Ritter-type result for random walk with drift conditioned to be positive. If that is indeed the case, it would follow (using the same arguments as in de Jong (2010)) that the Dickey-Fuller test should asymptotically indicate stationarity. In addition, such a Ritter-type result seems to imply that the KPSS statistic when applied to the exponential of the unit root process should converge in probability to 1/3, thereby asymptotically indicating stationarity as well. This remains an interesting and important topic for future research.

3 Simulations

Theorem 1 is easily illustrated with a simple simulation. A Matlab simulation program (available from the authors upon request) was used to generate simulation results for

$$Q_n(c, \alpha) = \sum_{t=1}^{n} \exp(c(y_t - M_n))$$

for various values of n, c, and α, and various distributions for i.i.d. ε_t that have a variance of 1. While Theorem 1 holds for any value of drift α and any value of the scaling parameter c, we should expect that the approximation will be poor for relatively low values of c. In that case after all, $y_t - M_n$ will be relatively small as well, and

$$\sum_{t=1}^{n} \exp(y_t - M_n) \approx \sum_{t=1}^{n} \exp(0) = n$$

and a large value for n will be needed in such a situation in order to achieve a good approximation to the limit distribution. Similarly for relatively large values of c,

$$\sum_{t=1}^{n} \exp(y_t - M_n) \approx \sum_{t=1}^{n} I(y_t = M_n) \approx 1.$$

In order to observe the convergence in distribution from the simulation, 10,000 replications were used everywhere to obtain the results. For n we used the values of 50, 100, 500, 1000, 5000 and 10,000 in all situations. For c, we chose 1, 2, and 5. For the distribution of ε_t, we used a standard normal, a uniform$[-\sqrt{3}, \sqrt{3}]$, and a Rademacher distribution (i.e., $P(\varepsilon_t = -1) = P(\varepsilon_t = 1) = 0.5$). Finally for the drift, we chose α =0.04 and 0.1.

Simulation results can be found in the tables of Appendix 2. For $\alpha = 0.04$, $c = 1$, the convergence in distribution appears to be relatively slow, especially for the uniform distribution case. For higher values of c and α, the convergence in distribution appears to be rapid, with a good approximation to the limit distribution being reached for $n = 1000$ to $n = 5000$. From the simulations it is clear that the limit distribution depends on the distribution of ε_t.

References

Apostol, T.M. (1973), *Mathematical Analysis*, 2nd edition. Addison–Wesley.

Borodin, A. N. and I.A. Ibragimov, (1995), Limit theorems for functionals of random walks, *Proceedings of the Steklov Institute of Mathematics 195*.

Burridge, P. and E. Guerre (1996), The limit distribution of level crossings of a random walk, and a simple unit root test, *Econometric Theory* 12, 705-723.

Corradi, V. (1995), Nonlinear transformation of integrated time series: a reconsideration?, *Journal of Time Series Analysis* 16, 539-549.

Davies, P.L. and W. Krämer (2003), The Dickey-Fuller test for exponential random walks, *Econometric Theory* 19, 865-877.

de Jong, R. (2010), Exponentials of unit root processes, *Working paper*.

Ermini, L. and D.F. Hendry (2008), Log incom vs. linear income: An application of the encompassing principle, *Oxford Bulletin of Economics and Statistics* 70, 807-827.

Ermini, L. and C.W.J. Granger (1993), Some generalizations on the algebra of I(1) processes, *Journal of Econometrics* 58, 369-384.

Feller, W. (1968), *An introduction to probability theory and its application*, Volume 2. New York: Wiley.

Breitung, J. and C. Gourieroux (1997), Rank tests for unit roots, *Journal of Econometrics* 81, 7-27.

Granger, C.W.J. and J. Hallmann (1991), Nonlinear transformations of the integrated time series, *Journal of Time Series Analysis* 12, 207-224.

Park, J.Y. and P.C.B. Phillips (1999), Asymptotics for nonlinear transformations of integrated time series, *Econometric Theory* 15, 269-298.

Ritter, G.A. (1981), Growth of random walks conditioned to stay positive, *Annals of Probability* 9, 699-704.

Appendix 1: Mathematical proofs

Proof of Lemma 1: Similarly to Davies and Krämer's argument of page 868[2], after applying integration by parts to the Rieman-Stieltjes integral (Theorem 7.6 in Apostol, p.144), we

[2]Davies and Krämer added a term to this formula. We thank Professor Davies for pointing this out to us.

have

$$
\begin{aligned}
&E\exp(y_t - M_n) = \int_{-\infty}^{\infty} \exp(-x) dF_{tn}(x) \\
&= \exp(-x)F_{tn}(x)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \exp(-x)F_{tn}(x)dx \\
&= \int_{-\infty}^{\infty} \exp(-x)F_{tn}(x)dx = \int_{0}^{\infty} \exp(-x)F_{tn}(x)dx
\end{aligned}
$$

where $F_{tn}(\cdot)$denotes the distribution function of $M_n - y_t$. $F_{tn}(z) = 0$ if $z < 0$ and for $z \geq 0$ we now have for $t \in \{1, ..., n-1\}$

$$
\begin{aligned}
&F_{tn}(z) = P(M_n - y_t \leq z) = P(\max_{1\leq s\leq n}(y_s - y_t) \leq z) \\
&= P(\max_{1\leq s\leq t-1}(y_s - y_t) \leq z)P(\max_{t+1\leq s\leq n}(y_s - y_t) \leq z) \\
&= P(\max_{1\leq s\leq t-1}(-y_s) \leq z)P(\max_{1\leq s\leq n-t} y_s \leq z) \\
&\leq P(\max_{1\leq s\leq n-t} y_s \leq z).
\end{aligned}
$$

Now if $z \leq \alpha t/2$, by the Markov inequality, for $t \in \{1, ..., n\}$,

$$
P(\max_{1\leq s\leq t} y_s \leq z) \leq P(t^{-1/2}(y_t - \alpha t) \leq t^{-1/2}(z - \alpha t)) \leq P(|t^{-1/2}(y_t - \alpha t)|^{2+\delta} \geq (\alpha t^{1/2}/2)^{2+\delta})
$$

$$
\leq t^{-1-\delta/2}\alpha^{-2-\delta}2^{2+\delta} \sup_{t\geq 1} E\left|t^{-1/2}(y_t - \alpha t)\right|^{2+\delta}
$$

$$
= t^{-1-\delta/2}\alpha^{-2-\delta}2^{2+\delta} \sup_{t\geq 1} E|t^{-1/2}\sum_{j=1}^{t} \varepsilon_j|^{2+\delta} \leq Ct^{-1-\delta/2},
$$

where $C = \alpha^{-2-\delta}2^{2+\delta} \sup_{t\geq 1} E|t^{-1/2}\sum_{j=1}^{t} \varepsilon_j|^{2+\delta} < \infty$. This holds because for $p = 2 + \delta$ we have

$$
\| t^{-1/2}\sum_{j=1}^{t} \varepsilon_j \|_p \leq C_p \| t^{-1/2}(\sum_{j=1}^{t} \varepsilon_j^2)^{1/2} \|_p = C_p(E|t^{-1}\sum_{j=1}^{t} \varepsilon_j^2|^{p/2})^{1/p}
$$

$$
= C_p(\| t^{-1}\sum_{j=1}^{t} \varepsilon_j^2 \|_{p/2})^{1/2} \leq C_p(\| \varepsilon_j^2 \|_{p/2})^{1/2} = C_p(E|\varepsilon_j|^p)^{1/p} < \infty.
$$

The first inequality is Burkholder's inequality, the last inequality is Jensen's, and we used the definition $\|Y\|_p = (E|Y|^p)^{1/p}$. This last result implies now that for $z \leq \alpha(n-t)/2$ and $t \in \{1, ..., n-1\}$,

$$F_{tn}(z) \leq P(\max_{1\leq s\leq n-t} y_s \leq z) \leq C(n-t)^{-1-\delta/2}.$$

Therefore, the proof is now complete by noting that

$$\begin{aligned}
ER_n &= \sum_{t=1}^{n} \int_0^\infty \exp(-x) F_{tn}(x) dx \\
&\leq 1 + \sum_{t=1}^{n-1} \int_0^\infty \exp(-x) F_{tn}(x) dx \\
&= 1 + \sum_{t=1}^{n-1} \int_0^{\alpha(n-t)/2} \exp(-x) F_{tn}(x) dx + \sum_{t=1}^{n-1} \int_{\alpha(n-t)/2}^\infty \exp(-x) F_{tn}(x) dx \\
&\leq 1 + \sum_{t=1}^{n-1} \int_0^{\alpha(n-t)/2} C(n-t)^{-p/2} \exp(-x) dx + \sum_{t=1}^{n-1} \int_{\alpha(n-t)/2}^\infty \exp(-x) dx \\
&\leq 1 + C \sum_{t=1}^{\infty} t^{-1-\delta/2} + \sum_{t=1}^{\infty} \exp(-\alpha t/2) < \infty.
\end{aligned}$$

□

Proof of Lemma 2: Note that u_k is nonincreasing and bounded from below by 0 because

$$\begin{aligned}
0 \leq u_k &= E(\exp(-r \sum_{t=1}^{k} \exp(-y_t)) I(\min_{1\leq s\leq k} y_s > 0)) \\
&= E(\exp(-r \sum_{t=1}^{k-1} \exp(-y_t) \exp(-r \exp(-y_k))) I(\min_{1\leq s\leq k-1} y_s > 0) I(y_k > 0)) \\
&\leq E(\exp(-r \sum_{t=1}^{k-1} \exp(-y_t)) I(\min_{1\leq s\leq k-1} y_s > 0)) = u_{k-1}.
\end{aligned}$$

Therefore, u_k converges to $u = \lim_{n\to\infty} u_n$.

Similarly, $\sum_{k=1}^{n} v_{n-k} = 1 + \sum_{k=1}^{n-1} v_k$ is increasing in n and bounded from above because

$$\begin{aligned}
0 \leq v_k &= E(\exp(-r\sum_{t=1}^{k}\exp(y_t))I(\max_{1\leq s\leq k} y_s \leq 0)) \\
&\leq EI(\max_{1\leq s\leq k} y_s \leq 0) = P(\max_{1\leq s\leq k} y_s \leq 0),
\end{aligned}$$

and from the proof of Lemma 1, $P(\max_{1\leq s\leq k} y_s \leq 0) \leq Ck^{-1-\frac{\delta}{2}}$. Therefore,

$$\sum_{k=1}^{n} v_{n-k} = 1 + \sum_{k=1}^{n-1} v_k \leq 1 + C\sum_{k=1}^{\infty} k^{-1-\frac{\delta}{2}} < \infty,$$

implying that $\sum_{k=1}^{n} v_{n-k}$ converges to $\lim_{n\to\infty}\sum_{k=1}^{n} v_{n-k} = v$.

$\square$

Proof of Theorem 1: First we need to show the convergence of $E\exp(-rR_n)$, i.e. the convergence of $\sum_{k=1}^{n} u_{k-1}v_{n-k}$. This follows by the convergence of u_n and $\sum_{k=1}^{n} v_{n-k}$ as $n \to \infty$, since for all integers $M > 1$ and $n > M$, because $u_k \geq 0$, $v_k \geq 0$, and u_k is decreasing,

$$\begin{aligned}
|\sum_{k=1}^{n}(u_{k-1}-u)v_{n-k}| &= \sum_{k=1}^{n}(u_{k-1}-u)v_{n-k} \\
&= \sum_{k=1}^{M}(u_{k-1}-u)v_{n-k} + \sum_{k=M+1}^{n}(u_{k-1}-u)v_{n-k} \\
&\leq (u_0-u)\sum_{k=1}^{M} v_{n-k} + (u_M-u)\sum_{k=M+1}^{n} v_{n-k} \\
&= (u_0-u)\sum_{k=n-M}^{n-1} v_k + (u_M-u)\sum_{k=0}^{n-M-1} v_k.
\end{aligned}$$

By making n approach infinity first and then making M approach infinity, it now follows that $\lim_{n\to\infty}\sum_{k=1}^{n} u_{k-1}v_{n-k} - u\lim_{n\to\infty}\sum_{k=1}^{n} v_k = 0$. Therefore $\sum_{k=1}^{n} u_{k-1}v_{n-k}$ converges to uv, which implies that $E\exp(-rR_n) \to \exp(-r)uv$.

By Feller's (1968) Theorem 2 of page 408, $\exp(-r)uv$ is the transform of a possibly defective distribution $F(\cdot)$, and the limit $F(\cdot)$ is not defective if $\exp(-r)uv \to 1$ as $r \downarrow 0$. Since

$$|\exp(-r)uv - 1| = \lim_{n\to\infty} |E\exp(-r\sum_{t=1}^{n}\exp(y_t - M_n)) - 1|$$
$$\leq |r| \sup_{n\geq 1} E\sum_{t=1}^{n}\exp(y_t - M_n)$$

because $1-\exp(-|x|) \leq |x|$ and $\sup_{n\geq 1} E\sum_{t=1}^{n}\exp(y_t - M_n) < \infty$ as was shown in Lemma 1, it follows that the limit distribution is not defective. $\square$

Appendix 2: Simulation results

Table 1: Simulated quantiles and mean of the distribution of $Q_n(1, 0.04)$; ε_t Stdnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.87	2.32	3.21	4.49	6.24	8.10	9.27	4.90
100	2.04	2.50	3.54	4.97	6.83	8.91	10.31	5.42
500	2.36	2.86	3.92	5.54	7.69	10.26	12.10	6.15
1000	2.42	2.92	3.98	5.70	7.90	10.58	12.51	6.31
5000	2.48	2.99	4.06	5.76	7.98	10.69	12.68	6.41
10000	2.48	2.96	3.98	5.63	7.98	10.58	12.47	6.33

Table 2: Simulated quantiles and mean of the distribution of $Q_n(2, 0.04)$; ε_t Stdnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.07	1.23	1.62	2.25	3.10	4.10	4.73	2.47
100	1.15	1.30	1.72	2.37	3.27	4.33	5.05	2.63
500	1.24	1.40	1.83	2.53	3.47	4.56	5.31	2.81
1000	1.25	1.42	1.85	2.54	3.53	4.70	5.58	2.86
5000	1.27	1.44	1.87	2.55	3.56	4.74	5.60	2.88
10000	1.27	1.44	1.85	2.55	3.53	4.70	5.54	2.86

Table 3: Simulated quantiles and mean of the distribution of $Q_n(5, 0.04)$; ε_t Stdnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.00	1.00	1.04	1.24	1.68	2.17	2.54	1.41
100	1.00	1.01	1.06	1.28	1.76	2.27	2.63	1.47
500	1.00	1.01	1.07	1.31	1.79	2.29	2.67	1.51
1000	1.00	1.02	1.08	1.32	1.81	2.33	2.72	1.53
5000	1.01	1.02	1.08	1.32	1.83	2.34	2.74	1.54
10000	1.01	1.02	1.08	1.33	1.80	2.33	2.73	1.53

Table 4: Simulated quantiles and mean of the distribution of $Q_n(1, 0.04)$; ε_t Uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.76	2.20	3.07	4.25	5.75	7.46	8.61	4.57
100	1.99	2.42	3.31	4.66	6.42	8.44	9.69	5.09
500	2.46	2.90	3.89	5.35	7.47	9.85	11.57	5.98
1000	2.50	2.99	4.00	5.56	7.74	10.14	11.89	6.18
5000	2.68	3.14	4.15	5.81	8.14	10.74	12.63	6.48
10000	2.74	3.21	4.22	5.94	8.21	10.94	12.73	6.60

Table 5: Simulated quantiles and mean of the distribution of $Q_n(2, 0.04)$; ε_t Uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.03	1.20	1.51	2.08	2.85	3.74	4.33	2.28
100	1.12	1.26	1.61	2.21	3.03	4.02	4.67	2.44
500	1.21	1.34	1.72	2.35	3.24	4.33	5.10	2.63
1000	1.26	1.40	1.77	2.43	3.31	4.40	5.18	2.70
5000	1.30	1.43	1.81	2.47	3.41	4.49	5.32	2.77
10000	1.30	1.44	1.81	2.47	3.41	4.53	5.30	2.77

Table 6: Simulated quantiles and mean of the distribution of $Q_n(5, 0.04)$; ε_t Uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.17	1.00	1.03	1.18	1.59	2.04	2.40	1.32
100	0.63	1.00	1.03	1.20	1.62	2.08	2.44	1.36
500	1.00	1.01	1.05	1.25	1.71	2.20	2.54	1.44
1000	1.00	1.01	1.05	1.26	1.72	2.21	2.55	1.46
5000	1.01	1.01	1.07	1.28	1.74	2.24	2.61	1.48
10000	1.01	1.01	1.06	1.28	1.72	2.21	2.62	1.48

Table 7: Simulated quantiles and mean of the distribution of $Q_n(1, 0.04)$; ε_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	2.02	2.30	2.99	4.06	5.49	7.01	8.00	4.41
100	2.19	2.48	3.22	4.46	6.04	7.86	9.07	4.87
500	2.38	2.72	3.54	4.91	6.86	8.98	10.64	5.50
1000	2.40	2.76	3.63	5.03	7.06	9.18	10.84	5.62
5000	2.42	2.79	3.68	5.09	7.10	9.35	11.14	5.70
10000	2.43	2.78	3.67	5.09	7.06	9.47	11.19	5.72

Table 8: Simulated quantiles and mean of the distribution of $Q_n(2, 0.04)$; ε_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.21	1.31	1.44	1.98	2.67	3.43	3.98	2.19
100	1.27	1.33	1.50	2.08	2.81	3.63	4.18	2.29
500	1.32	1.35	1.54	2.20	2.97	3.83	4.44	2.40
1000	1.32	1.35	1.55	2.24	2.98	3.87	4.54	2.43
5000	1.32	1.36	1.57	2.27	3.08	4.01	4.64	2.48
10000	1.32	1.36	1.58	2.27	3.04	3.97	4.61	2.47

Table 9: Simulated quantiles and mean of the distribution of $Q_n(5, 0.04)$; ε_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.01	1.01	1.02	1.07	1.69	2.01	2.20	1.35
100	1.01	1.01	1.02	1.10	1.69	2.06	2.25	1.37
500	1.01	1.01	1.02	1.12	1.69	2.14	2.35	1.40
1000	1.01	1.01	1.02	1.13	1.69	2.14	2.33	1.40
5000	1.01	1.01	1.02	1.14	1.69	2.13	2.33	1.40
10000	1.01	1.01	1.02	1.14	1.69	2.14	2.36	1.40

Table 10: Simulated quantiles and mean of the distribution of $Q_n(1, 0.1)$; ε_t Stdnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.83	2.22	3.08	4.36	6.00	7.79	8.95	4.75
100	2.00	2.40	3.29	4.69	6.54	8.63	10.18	5.17
500	2.09	2.50	3.45	4.85	6.87	9.27	10.91	5.48
1000	2.09	2.53	3.45	4.89	6.88	9.22	10.94	5.49
5000	2.08	2.52	3.49	4.91	6.90	9.20	10.97	5.50
10000	2.09	2.53	3.49	4.94	6.92	9.33	10.90	5.52

Table 11: Simulated quantiles and mean of the distribution of $Q_n(2, 0.1)$; ε_t Stdnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.11	1.24	1.61	2.23	3.04	4.01	4.68	2.46
100	1.15	1.29	1.68	2.31	3.18	4.18	4.88	2.57
500	1.19	1.35	1.75	2.41	3.33	4.40	5.21	2.69
1000	1.18	1.33	1.73	2.38	3.31	4.38	5.19	2.67
5000	1.20	1.34	1.74	2.39	3.30	4.37	5.11	2.67
10000	1.20	1.34	1.72	2.36	3.26	4.41	5.14	2.66

Table 12: Simulated quantiles and mean of the distribution of $Q_n(5, 0.1)$; ε_t Stdnormal

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.00	1.00	1.05	1.25	1.70	2.16	2.51	1.43
100	1.00	1.01	1.06	1.28	1.74	2.24	2.61	1.47
500	1.00	1.01	1.06	1.30	1.76	2.26	2.61	1.49
1000	1.00	1.01	1.07	1.29	1.76	2.24	2.61	1.49
5000	1.00	1.01	1.07	1.29	1.76	2.25	2.63	1.50
10000	1.00	1.01	1.06	1.30	1.76	2.28	2.68	1.50

Table 13: Simulated quantiles and mean of the distribution of $Q_n(1, 0.1)$; ε_t Uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.80	2.22	3.05	4.26	5.82	7.44	8.59	4.60
100	2.01	2.46	3.36	4.69	6.46	8.42	9.85	5.15
500	2.43	2.83	3.79	5.32	7.40	9.94	11.58	5.94
1000	2.50	2.96	4.00	5.60	7.76	10.30	12.04	6.21
5000	2.65	3.12	4.14	5.81	8.15	10.79	12.77	6.49
10000	2.71	3.20	4.23	5.88	8.24	10.94	12.98	6.60

Table 14: Simulated quantiles and mean of the distribution of $Q_n(2, 0.1)$; ε_t Uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.05	1.20	1.52	2.10	2.88	3.75	4.39	2.30
100	1.11	1.26	1.61	2.21	3.05	4.03	4.77	2.45
500	1.22	1.36	1.74	2.37	3.29	4.38	5.16	2.66
1000	1.25	1.38	1.77	2.40	3.35	4.42	5.21	2.71
5000	1.28	1.43	1.82	2.47	3.40	4.51	5.31	2.77
10000	1.29	1.43	1.81	2.48	3.43	4.57	5.38	2.79

Table 15: Simulated quantiles and mean of the distribution of $Q_n(5, 0.1)$; ε_t Uniform

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	0.27	1.00	1.02	1.18	1.59	2.04	2.37	1.33
100	1.00	1.00	1.03	1.21	1.64	2.09	2.42	1.37
500	1.00	1.01	1.05	1.24	1.67	2.17	2.56	1.43
1000	1.00	1.01	1.06	1.26	1.71	2.20	2.60	1.45
5000	1.00	1.01	1.06	1.27	1.74	2.21	2.58	1.47
10000	1.00	1.01	1.07	1.28	1.73	2.22	2.59	1.48

Table 16: Simulated quantiles and mean of the distribution of $Q_n(1, 0.1)$; ε_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.97	2.25	2.88	3.88	5.21	6.65	7.67	4.22
100	1.99	2.31	3.00	4.13	5.58	7.38	8.58	4.54
500	2.04	2.42	3.18	4.38	6.06	7.97	9.41	4.88
1000	2.02	2.37	3.11	4.28	5.96	8.00	9.41	4.81
5000	1.99	2.35	3.11	4.30	6.02	7.99	9.37	4.83
10000	2.05	2.40	3.15	4.35	6.01	7.93	9.37	4.84

Table 17: Simulated quantiles and mean of the distribution of $Q_n(2, 0.1)$; ε_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.20	1.30	1.42	1.91	2.48	3.14	3.62	2.07
100	1.21	1.32	1.46	1.96	2.56	3.30	3.82	2.15
500	1.21	1.32	1.46	2.02	2.64	3.42	4.03	2.20
1000	1.21	1.32	1.47	2.02	2.64	3.39	3.98	2.19
5000	1.21	1.33	1.47	2.01	2.65	3.41	3.98	2.20
10000	1.21	1.33	1.47	2.03	2.65	3.45	3.96	2.21

Table 18: Simulated quantiles and mean of the distribution of $Q_n(5, 0.1)$; ε_t Rademacher

n	0.05	0.10	0.25	0.5	0.75	0.90	0.95	mean
50	1.00	1.02	1.02	1.10	1.39	1.62	1.85	1.25
100	1.00	1.02	1.02	1.10	1.40	1.63	1.91	1.26
500	1.01	1.02	1.02	1.12	1.41	1.66	1.98	1.27
1000	1.01	1.02	1.02	1.11	1.41	1.65	1.92	1.26
5000	1.01	1.02	1.02	1.11	1.41	1.65	1.94	1.26
10000	1.01	1.02	1.02	1.13	1.41	1.66	1.98	1.27